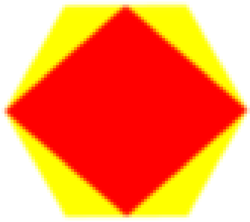
SOLUTIONS SEPTEMBER 2020

High school problems. 16-18 years. Authors: Collective "Concurso de Primavera". Madrid's community. XXI Concurso de Primavera. 2017.

https://www.concursoprimavera.es/#libros

September 1: The area of the rhombus inscribed in the regular hexagon is 24 cm^2. Find the area of the regular hexagon



Solution: Let r be the radius of the circle circumscribed to the regular hexagon (and equal to the side of the hexagon). Then the rhombus decomposes into four right triangles of legs r and $\frac{r\sqrt{3}}{2}$. Therefore:

$$A_{rhombus} = 4 \cdot \frac{r \cdot \frac{r \cdot \sqrt{3}}{2}}{2} = r^2 \cdot \sqrt{3} = 24$$

Finally, we can decompose the hexagon into six equilateral triangles with side r. Therefore:

$$\mathbf{A_{hexagon}} = 6 \cdot A_\Delta = 6 \cdot \frac{r \cdot \frac{r \cdot \sqrt{3}}{2}}{2} = \frac{6}{4} \cdot r^2 \cdot \sqrt{3} = \frac{6}{4} \cdot 24 = \mathbf{36\ cm^2}$$

September 2: Calculate the digit of the units of the sum of all the products of eight in eight of the numbers from 1 to 9

Solution: The sum of the statement is:

$$\sum_{k=1}^{9} \frac{9!}{k}$$

Each addend of the previous sum contains, at least, a factor 2 and a factor 5 (and therefore the number of the units of each of them is 0) except the corresponding addend to k = 5 which is: 9 · 8 · 7 · 6 · 4 · 3 · 2. Grouping two by two the factors of the previous product and taking into account only the figure of the units of each product:

$$9\cdot8\cdot7\cdot6\cdot4\cdot3\cdot2 \rightarrow 2\cdot2\cdot2\cdot2 \rightarrow 4\cdot4 \rightarrow 6$$

September 3: If a, b and c are positive integers with

abc + ab + ac + bc + a + b + c = 104

What is the value of $a^2+b^2+c^2$?

Solution: We have successively:

$$abc + ab + ac + bc + a + b + c = ab(c+1) + b(c+1) + a(c+1) + c = 104$$

Adding 1 to both sides, we get:

$$ab(c+1) + b(c+1) + a(c+1) + c + 1 = 105$$

$$(ab + b + a + 1)(c+1) = 105 = 3 \cdot 5 \cdot 7$$

$$(a+1)(b+1)(c+1) = 105 = 3 \cdot 5 \cdot 7$$

$$a \cdot b \cdot c = 2 \cdot 4 \cdot 6$$

$$a^2 + b^2 + c^2 = 2^2 + 4^2 + 6^2 = 56$$

September 4: How many points on the circumference $x^2 + y^2 = 50$ have at least one of the integer coordinates?

Solution: There are 30 points where x is an integer: from $x = -7$ to $x = 7$ (including $x = 0$). There are 15 whole abscissas. We consider the double because with the same abscissa there are two points with opposite ordinates. There are also 30 points with and integer. But there are 12 points that have both x and y integers and that have been counted twice: $(\pm5; \pm5)$; $(\pm1; \pm7)$; $(\pm7; \pm1)$. So the answer is: 30 + 30 - 12 = 48

September 5: If

$$\left.\begin{array}{r} \mathbf{tg\,x + tg\,y = 25} \\ \mathbf{cotg\,x + cotg\,y = 30} \end{array}\right\}$$

What is the value of tag (x+y)?

Solution: From the trigonometric relations of the angle addition and subtraction we have:

$$\mathrm{tg}(x+y) = \frac{\mathrm{tgx} + \mathrm{tgy}}{1 - \mathrm{tgx} \cdot \mathrm{tgy}} = \frac{25}{1 - \mathrm{tgx} \cdot \mathrm{tgy}}$$

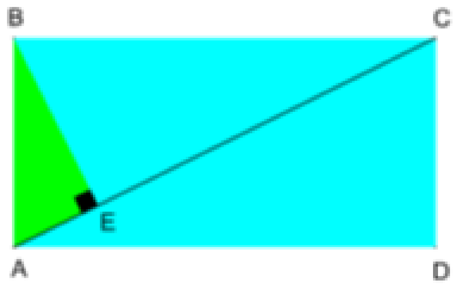
From the second equation:

$$30 = \frac{1}{\mathrm{tgx}} + \frac{1}{\mathrm{tgy}} = \frac{\mathrm{tgy} + \mathrm{tgx}}{\mathrm{tgx} \cdot \mathrm{tgy}} = \frac{25}{\mathrm{tgx} \cdot \mathrm{tgy}} \quad \Rightarrow \quad \mathrm{tgx} \cdot \mathrm{tgy} = \frac{25}{30} = \frac{5}{6}$$

By last:

$$\mathrm{tg}(x+y) = \frac{25}{1 - \mathrm{tgx} \cdot \mathrm{tgy}} = \frac{25}{1 - \frac{5}{6}} = 150$$

September 7-8: The sides of the rectangle in the figure are one twice the other. If BE⊥AC. What is the ratio of the area of triangle ΔABE to the area of rectangle ABCD?



Solution: If we consider AB = 1 then AD = 2 and AC = $\sqrt{5}$. Furthermore, we have: $\Delta AEB \cong \Delta ABC$ (since both are rectangles and have the angle at A equal). Therefore

$$\frac{1}{\sqrt{5}} = \frac{BE}{2} \Rightarrow BE = \frac{2}{\sqrt{5}} \Rightarrow AE = \sqrt{1^2 - BE^2} = \sqrt{1 - \frac{4}{5}} = \frac{1}{\sqrt{5}} \Rightarrow A_{\Delta ABE} = \frac{1}{2} \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{1}{5}$$

$$\frac{A_{\Delta ABE}}{A_{ABCD}} = \frac{\frac{1}{5}}{2 \cdot 1} = \frac{1}{10}$$

September 9: Let z = 9 + bi with b> 0. If the imaginary parts of z^2 and z^3 are equal, what is the value of b?

Solution: We have:

$$z^2 = (9 + bi)^2 = 81 - b^2 + 18bi$$

$$z^3 = (9 + bi)^3 = 729 - 27b^2 + (162b + 81b - b^3)i$$

Since the imaginary parts of both complexes are equal, we have:

$$18b = 243b - b^3 \quad \Rightarrow \quad 0 = -b^3 + 225b \quad \Rightarrow \quad b \in \{0, -15, 15\}$$

And since b > 0, we have b = 15

September 10: Solve in ℕ

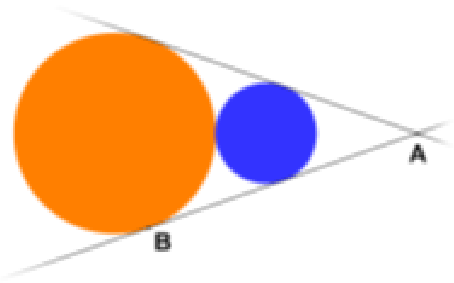
$$\left.\begin{array}{r} \mathbf{p + q \le 100} \\ \mathbf{\dfrac{p + q^{-1}}{p^{-1} + q} = 17} \end{array}\right\}$$

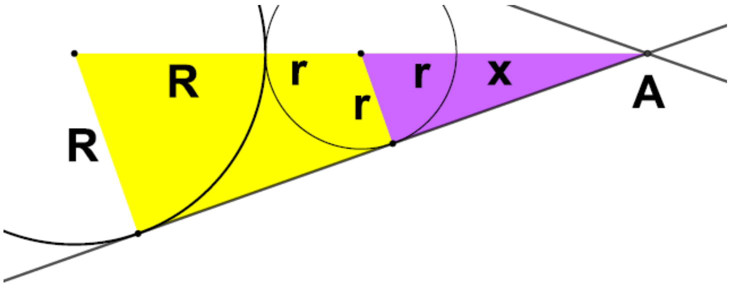
Solution: From the equation we have:

$$\frac{p + q^{-1}}{p^{-1} + q} = 17; \quad \frac{p + \frac{1}{q}}{\frac{1}{p} + q} = 17; \quad \frac{\frac{pq + 1}{q}}{\frac{1 + pq}{p}} = 17; \quad \frac{p}{q} = 17; \quad p = 17q$$

And since p + q $\le$ 100, we have 18q $\le$ 100. Hence q$\in${1, 2, 3, 4, 5} and p = 17q $\in${17, 34, 51, 68, 85}. In short: (q, p) $\in${(1,17); (2, 34); (3, 51); (4, 68); (5, 85)}

September 11-12: In the figure there are two circles tangent to each other and tangent to two lines that intersect at A. If B is a point of tangency, find AB as a function of the radii of the circles





Solution: The triangles in the attached figure are similar (since they are rectangles since the radius and tangent are perpendicular and have the angle in A common), therefore:

$$\frac{x+r}{r} = \frac{x+2r+R}{R}$$

From where, we get:

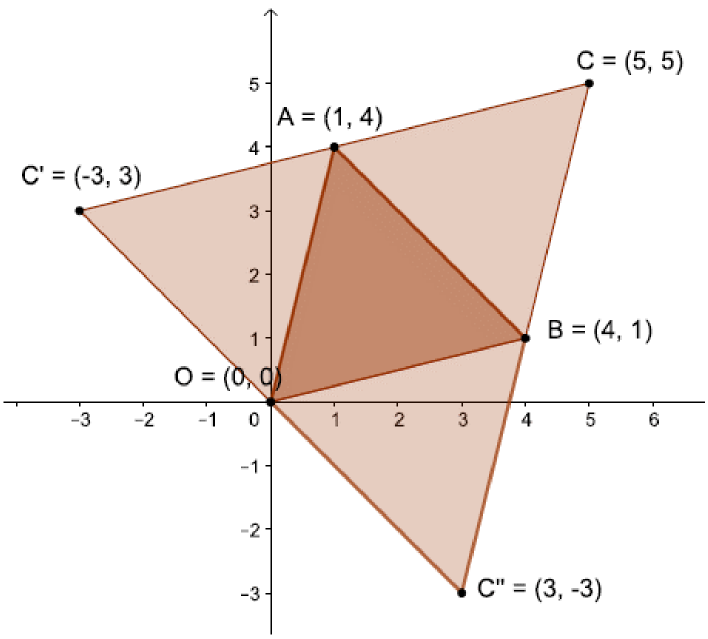
$$x = \frac{2r^2}{R-r}$$

By last:

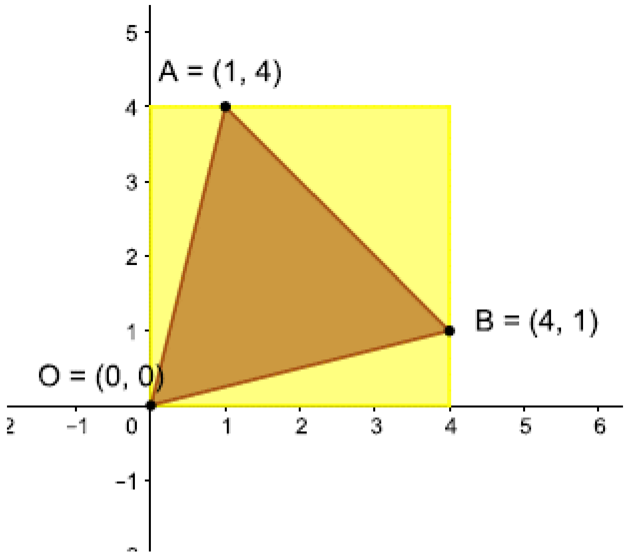
$$AB = d(A,B) = \sqrt{(R+2r+x)^2 - R^2} = \sqrt{\left(R+2r+\frac{2r^2}{R-r}\right)^2 - R^2} = \cdots = \frac{2R\sqrt{Rr}}{R-r}$$

September 14: Three vertices of a parallelogram are the points O (0,0); A (1,4) and B (4,1). Find the area of the parallelogram.

Solution: Three parallelograms are possible: OACB, OBAC' and OABC''. They all have the same area which is twice the area of the triangle ΔOAB



To calculate the area of the triangle ΔOAB, we have:



$$A_{\Delta ABO} = 4^2 - \frac{1}{2}(4 \cdot 1) - \frac{1}{2}(4 \cdot 1) - \frac{1}{2}(3 \cdot 3) = \frac{15}{2} \quad \Rightarrow \quad A = \frac{15}{2} \cdot 2 = 15$$

September 15: Solve in $\mathbb{N}$

$$\left.\begin{array}{c} 1 \leq a \leq 10 \\ a^{2020} + a^{2021} = \dot{\hat{5}} \end{array}\right\}$$

Solution: We will see if the second equation is satisfied or not for each of the possible values of a, which according to the first double inequality are, $a \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We will use the fact that $a^{2020} + a^{2021} = a^{2020} \cdot (1 + a)$

a = 1 it is not a solution because, for a = 1, we have:

$1^{2020} \cdot (1 + 1) = 1 \cdot 2 = 2$, which is not a multiple of 5

a = 2 it is not a solution because, for a = 2 we have:

$2^{2020} \cdot 3$ it is not a multiple of 5 since 2^{2020} ends in 6 and not in 0 or 5, since

n	2^n ends in
1, 5, 9, …… $\equiv 1(4)$	2
2, 6, 10,… $\equiv 2(4)$	4
3, 7, 11,… $\equiv 3(4)$	8
4, 8, 12,… $\equiv 0(4)$	6

a = 3 it is not a solution because, for a = 3 we have:

$3^{2020} \cdot 4$ it is not a multiple of 5 since 3^{2020} ends in 1 and not in 0 or 5, since

n	3^n ends in
1, 5, 9, …… $\equiv 1(4)$	3
2, 6, 10,… $\equiv 2(4)$	9
3, 7, 11,… $\equiv 3(4)$	7
4, 8, 12,… $\equiv 0(4)$	1

a = 4 is solution because, for a = 4 we have:

$$a^{2020} + a^{2021} = a^{2020} \cdot (1 + a) = 4^{2020} \cdot 5 \text{ which is a multiple of 5}$$

a = 5 is solution because, for a = 5 we have:

$$a^{2020} + a^{2021} = a^{2020} \cdot (1 + a) = 5^{2020} \cdot 6 \text{ which is a multiple of 5}$$

a = 6 it is not a solution because, for a = 6 we have:

$6^{2020} \cdot 7$ it is not a multiple of 5 since 6^{2020} ends in 6 and not in 0 or 5

a = 7 it is not a solution because, for a = 7 we have:

$7^{2020} \cdot 8$ it is not a multiple of 5 since 7^{2020} ends in 1 and not in 0 or 5, since

n	7^n ends in
1, 5, 9, …… $\equiv 1(4)$	7
2, 6, 10,… $\equiv 2(4)$	9
3, 7, 11,… $\equiv 3(4)$	3
4, 8, 12,… $\equiv 0(4)$	1

a = 8 it is not a solution because, for a = 8 we have:

$8^{2020} \cdot 9$ it is not a multiple of 5 since 8^{2020} ends in 6 and not in 0 or 5, since

n	8^n ends in
1, 5, 9, …… $\equiv 1(4)$	8
2, 6, 10,… $\equiv 2(4)$	4
3, 7, 11,… $\equiv 3(4)$	2
4, 8, 12,… $\equiv 0(4)$	6

a = 9 is solution because, for a = 9 we have:

$$a^{2020} + a^{2021} = a^{2020} \cdot (1 + a) = 9^{2020} \cdot 10 \text{ which is a multiple of 5, since it ends in 0}$$

a = 10 is a solution because, for a = 10 we have:

$$a^{2020} + a^{2021} = a^{2020} \cdot (1 + a) = 10^{2020} \cdot 11 \text{ which is a multiple of 5, since it is a multiple of 10}$$

In total, the solutions are a = 4, a = 5, a = 9 y a = 10

September 16: The ordinates at the origin of three parallel lines are 2, 3, and 4. The sum of the abscissa of the intersection points of the lines with the X-axis is - 36, what is the slope of the three lines?

Solution: Let

$$y = mx + 2$$
$$y = mx + 3$$
$$y = mx + 4$$

the lines of the statement. The abscissa of the intersection points of the lines with the axis OX are

$$\frac{-2}{m};\ \frac{-3}{m};\ \frac{-4}{m}$$

And then

$$\frac{-2}{m}+\frac{-3}{m}+\frac{-4}{m}=-36;\ \ \frac{-9}{m}=-36;\ \ m=\frac{-9}{-36}=\frac{1}{4}$$

September 17-18: Suppose eight envelopes numbered from 1 to 8 and eight cards also numbered from 1 to 8. How many ways can the cards be distributed, one in each envelope, so that none of the cards 1, 2 and 3 are in the envelope with the same number?

Solution: Without restrictions, there would be 8! ways to distribute the cards. We will have to remove those where card 1 (7!), card 2 (7!) and card 3 (7!) are correct; but we have removed more, those where 1 and 2 are good, which we have counted twice (6!), 1 and 3 (6!) and 2 and 3 (6!); but now we have counted more, we must remove the cases in which 1, 2 and 3 are correct (5!). So, the requested number is:

$$8! - 3\cdot 7! + 3\cdot 6! - 5! = 5!\cdot(8\cdot 7\cdot 6 - 3\cdot 7\cdot 6 + 3\cdot 6 - 1) = 5!\cdot 217 = 27240$$

September 19: How many integers between 10 and 1000 verify that the sum of its digits is 3?

Solution: You just have to be a bit tidy: The digits that we can use are: 0, 1, 2 and 3.

With two digits we have: 12, 21 and 30.

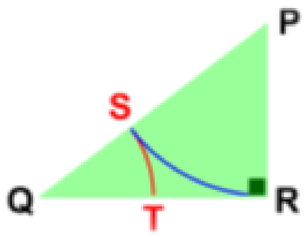
Of three digits we will have: 102, 111, 120, 201, 210 and 300.

In total 9 numbers.

September 21: How many positive three-digit integers have no digits other than 7, 8, or 9?

Solution: They are the variations with repetition of three elements (the number 7, the number 8 and the number 9) taken three by three, that is, $3^3 = 27$.

September 22-23: Let ΔPRQ be a right triangle at R. The circle with centre P and radius PR intersects PQ at S and the circle with centre Q and radius QS intersects QR at T. If T is the midpoint of QR, find QS / SP



Solution: We will have, successively:

$$QP^2 = (QS + SP)^2 = QR^2 + PR^2;\quad QS^2 + SP^2 + 2\cdot QS\cdot SP = 4\cdot QS^2 + PR^2$$

$$SP^2 + 2\cdot QS\cdot SP = 3\cdot QS^2 + PR^2,\ \ SP^2 + 2\cdot QS\cdot SP = 3\cdot QS^2 + SP^2,$$

$$2\cdot QS\cdot SP = 3\cdot QS^2,\ \ 2\cdot SP = 3\cdot QS\ \ \Rightarrow\ \ \frac{QS}{SP}=\frac{2}{3}$$

September 24: Find the pairs of integers (x, y) with $0 \le x \le y$ that satisfy:

$$5x^2 - 4xy + 2x + y^2 = 624$$

Solution: Regarding the equation, we have:

$$5x^2 - 4xy + 2x + y^2 = 624; \;\; 4x^2 - 4xy + y^2 + x^2 + 2x = 624$$

$$4x^2 - 4xy + y^2 + x^2 + 2x + 1 = 624 + 1; \;\; (2x - y)^2 + (x + 1)^2 = 625 = 25^2$$

Making m = 2x - y and n = x + 1, we have that $m,n \in \mathbb{Z}$ (since x and y are integers) and also:

$$m^2 + n^2 = 25^2$$

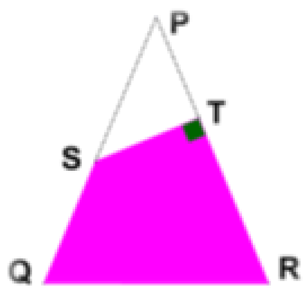
Since (15, 20, 25) and (7, 24, 25) are the only Pythagorean triples of hypotenuse 25 and $n = x + 1 \ge 1$ (since $x \ge 0$), the only integer solutions for m and n (with $n \ge 1$) they appear in the first two columns of the following table. The other columns find x and y y if the restriction $x \le y$ is satisfied

m	n	**x**	**y**	¿0 ≤ x ≤ y?
0	25	**24**	**48**	Yes
25	0	-1	-26	No
20	15	14	8	No
-20	15	**14**	**48**	Yes
15	20	**19**	**23**	Yes
-15	20	**19**	**53**	Yes
7	24	**23**	**39**	Yes
-7	24	**23**	**53**	Yes
24	7	6	-12	No
-24	7	**6**	**36**	Si

For example, for the first row we have:

$$\begin{cases} 2x - y = 0 \\ x + 1 = 25 \end{cases} \Rightarrow \begin{cases} y = 2 \cdot 24 = 48 \\ x = 24 \end{cases}$$

September 25-26: In an isosceles triangle ΔPQR, PQ = PR and QR = 300. On the side PR we take T and on the side PQ we take S such that $TS \perp TR$. If ST = 120, TR = 271 and QS = 221, find the area of quadrilateral STRQ



Solution: Since PQ = PR, if PT = x, then PS = x + 50 and the dimensions of the legs of the right triangle ΔPTS are x and 120 and the hypotenuse x + 50, therefore:

$$(x+50)^2 = x^2 + 120^2 \;\;\Rightarrow\;\; x = 119 \text{ and } A_{\Delta PTS} = \frac{120 \cdot 119}{2} = 7140$$

On the other hand, let's calculate the area of the isosceles triangle ΔPQR, for this we calculate the height on the side QR:

$$\sqrt{390^2 - 150^2} = 360$$

Therefore, the area will be:

$$\frac{300 \cdot 360}{2} = 54000$$

The requested area is:

54000 – 7140 = 46860

September 28: Let a, b and c be different integers that satisfy a·b·c = 17955; a, b and c (and in this order) are in arithmetic progression; 3a + b, 3b + c and 3c + a (and in this order) are in geometric progression. Calculate a, b and c

Solution: If a, b and c are in arithmetic progression, we can put: a = b - d and c = b + d: So, that 3a + b = 4b - 3d; 3b + c = 4b + d and 3c + a = 4b + 2d, and like these, they are in geometric progression:

$$(4b+d)^2 = (4b-3d)\cdot(4b+2d) \;\;\Rightarrow\;\; 16b^2 + 8bd + d^2 = 16b^2 - 4bd - 6d^2$$

$$12bd = -7d^2$$

Since d ≠ 0 (well, otherwise a = b = c and 17955 is not a perfect cube), it follows that:

$$d = -\frac{12b}{7}; \quad \Rightarrow \quad a = \frac{19b}{7}, c = -\frac{5b}{7}$$

And since a·b·c = 17955, we have:

$$\frac{19b}{7} \cdot b \cdot \frac{-5b}{7} = 17955 \;\;\Rightarrow\;\; b^3 = -9261 \;\;\Rightarrow\;\; b = -21 \;\;\Rightarrow a = -57 \text{ y } c = 15$$

September 29: Let a, b and c be three numbers in geometric progression. Find them, if their sum is 114 and their product is 46656.

Solution: Being a, b and c in geometric progression we have:

$$a = \frac{b}{r} \text{ and } c = br \;\;\Rightarrow\;\; a \cdot b \cdot c = b^3 = 46656 \;\;\Rightarrow\;\; b = 36$$

Therefore: a + 36 + c = 114 and therefore: a + c = 78

$$\frac{36}{r} + 36r = 78;\;\; 6r^2 - 13r + 6 = 0,\;\; r = \frac{3}{2} \text{ or } r = \frac{2}{3}$$

The numbers that are in geometric progression are: 24, 36, 54 or 54, 36, 24

<u>September 30:</u> Find the numerical value of $\mathbf{x^2 + y^2}$ knowing that:

$$\left.\begin{array}{r}\mathbf{x^2 = 8x + y}\\ \mathbf{y^2 = 8y + x}\\ \mathbf{x \neq y}\end{array}\right\}$$

<u>Solution 1 (by brute force):</u> From the first equation we have y = x^2 - 8x and substituting y in the second:

$$(x^2 - 8x)^2 = 8(x^2 - 8x) + x \;\Rightarrow\; x^4 - 16x^3 + 56x^2 + 63x = 0$$

If x = 0, then y (= 0^2 - 8·0) = 0 which contradicts the third condition of the system. Then x ≠ 0, and then:

$$x^3 - 16x^2 + 56x + 63 = 0$$

	1	-16	56	63
9		9	-63	63
	1	-7	-7	0

$$(x - 9) \cdot (x^2 - 7x - 7) = 0$$

If x = 9, then y (= 9^2 – 8·9) = 9 which contradicts the third condition of the system. Then x ≠ 9.

$$(x^2 - 7x - 7) = 0 \;\Rightarrow \begin{cases} x_1 = \dfrac{7+\sqrt{77}}{2} & \Rightarrow\; y_1 = \dfrac{7-\sqrt{77}}{2} \\ x_2 = \dfrac{7-\sqrt{77}}{2} & \Rightarrow\; y_2 = \dfrac{7+\sqrt{77}}{2} \end{cases}$$

Therefore:

$$x^2 + y^2 = \left(\frac{7+\sqrt{77}}{2}\right)^2 + \left(\frac{7-\sqrt{77}}{2}\right)^2 = 63$$

<u>Solution 2:</u> Adding the first two equations, we have:

$$x^2 + y^2 = 9 \cdot (x + y) \; (*)$$

Subtracting the first two equations, we have:

$$x^2 - y^2 = 7 \cdot (x - y) \;\Rightarrow\; (x + y) \cdot (x - y) = 7(x - y)$$

And since x ≠ y, it must be x + y = 7 and substituting in (*)

$$x^2 + y^2 = 9 \cdot (x + y) = 9 \cdot 7 = 63$$