SOLUTIONS OCTOBER 2020

PROBLEMS FOR 3ESO, 4ESO AND HIGH SCHOOL. AUTHOR: MARIO MESTRE. INS "PONT DE SUERT". 14-18 YEARS

October 1: Find k so that the roots of $3x^2+5x-k=0$ are differentiated into two units.

Solution: Let's remember Vietà's formulas: If x_1 and x_2 are the solutions of $ax^2 + bx + c = 0$ then:

$$S = x_1 + x_2 = -\frac{b}{a} \qquad P = x_1 \cdot x_2 = \frac{c}{a}$$

In our case we will have:

$$S = x_1 + x_1 + 2 = 2x_1 + 2 = -\frac{5}{3} \Rightarrow x_1 = \frac{-\frac{5}{3} - 2}{2} = -\frac{11}{6} \Rightarrow x_2 = x_1 + 2 = -\frac{11}{6} + 2 = \frac{1}{6}$$

And finally:

$$P = x_1 \cdot x_2 = -\frac{11}{6} \cdot \frac{1}{6} = -\frac{11}{36} = -\frac{k}{3} \Rightarrow k = \frac{11}{12}$$

October 2: Find the values of m that make

$$mx^2 - (m + 3)x + 2=0$$

have two opposite real roots

Solution: If x_1 and x_2 are the roots of the polynomial and are opposite:

$$S = 0 = \frac{m+3}{m} \Rightarrow m = -3$$

October 3: Calculate p and q so that the roots of $x^2+px+q=0$ are D and 1-D, where D is the discriminant of the equation.

Solution: We will have that $D = p^2 - 4q$. Therefore, the roots of the polynomial must be

$$x_1 = p^2 - 4q \ ; \ x_2 = 1 + 4q - p^2$$

Applying the Vietà formulas:

$$1 = S = x_1 + x_2 = -p \Rightarrow p = -1 \Rightarrow x_1 = 1 - 4q; \ x_2 = 4q$$

$$x_1 \cdot x_2 = q = (1 - 4q) \cdot 4q \Rightarrow \begin{cases} q = 0 \\ q = \frac{3}{16} \end{cases}$$

Therefore $(p, q) \in \{(-1, 0); (-1, 3/16)\}$

October 5: Let the equation be $x^2+px+q=0$. Find p and q so that p and q are the solutions of the equation.

Solution: We will have when applying the Vietà formulas:

$$\begin{cases} S = p + q = -p \\ P = p \cdot q = q \end{cases}$$

From the second equation:

$$q \cdot (p-1) = 0 \;\Rightarrow\; \begin{cases} q = 0 \\ p = 1 \end{cases}$$

And substituting in the first one: If q = 0, then p = 0. If p = 1, then q = -2.

The solutions are: (p, q) $\in$ {(0, 0); (1, -2)}

October 6: If r and s are the solutions of the equation x^2-17x+13 = 0, calculate the value of $r^3 + s^3$

Solution: We have, if S = r + s and P = r·s:

$$(r+s)^3 = r^3 + 3r^2s + 3rs^2 + s^3 \;\Rightarrow\; r^3 + s^3 = S^3 - 3 \cdot P \cdot S$$

Applying the Vietà formulas:

$$S = r + s = 17; \quad P = r \cdot s = 13$$

From where:

$$r^3 + s^3 = S^3 - 3 \cdot P \cdot S = 17^3 - 3 \cdot 17 \cdot 13 = 4250$$

October 7: If a and b are the roots of x^2-2x-143 = 0, find the value of

$$\frac{\mathbf{1}}{\mathbf{a}} + \frac{\mathbf{1}}{\mathbf{b}}$$

Solution: We have, if S = a + b and P = a·b what:

$$\frac{1}{a} + \frac{1}{b} = \frac{b+a}{a \cdot b} = \frac{S}{P}$$

Applying the Vietà formulas:

$$S = 2; \;\; P = -143$$

And, at last:

$$\frac{1}{a} + \frac{1}{b} = \frac{S}{P} = -\frac{143}{2}$$

October 8-9: Let P(x) = (x+1)·(x-8) + m.

a) Are there values of m for which P(x) > 0 $\forall x \in \mathbb{R}$?.
b) Are there values of m for which P(x) < 0 $\forall x \in \mathbb{R}$?.
c) If a and b are the roots of P (x), find m so that $a^2+b^2=1$.

Solution: We have:

$$P(x) = (x+1)\cdot(x-8)+m = x^2 - 7x - 8 + m$$

Therefore, its graphical representation is a parabola directed upwards, since its main coefficient is positive, with vertex (minimum of the graph)

$$x_V = \frac{-b}{2a} = \frac{7}{2}; \quad y_V = \left(\frac{7}{2} + 1\right) \cdot \left(\frac{7}{2} - 8\right) + m = -\frac{81}{4} + m$$

a) The polynomial will be positive if the y of the vertex is positive, that is, iff

$$-\frac{81}{4} + m > 0 \;\Leftrightarrow\; m > \frac{81}{4}$$

b) There are no values of m for which $P(x) < 0\ \forall x \in \mathbb{R}$, since the parabola is directed upwards (for any value of m) and therefore there are positive values of the parabola (for any value of m).

c) If S = a + b and P = a·b, we have:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \Rightarrow \quad a^2 + b^2 = S^2 - 2P$$

For Vietà formulae: S = 7 and P =m – 8. Therefore:

$$1 = a^2 + b^2 = S^2 - 2P = 7^2 - 2m + 16 \ \Rightarrow \ m = 32$$

October 10: If α and β are the roots of:

$$\mathbf{0 = x^2 - 4x + 22}$$

calculate the value of

$$\boldsymbol{\alpha^3 + \alpha^2 + \alpha + \beta^3 + \beta^2 + \beta}$$

Solution: If S = α + β and P = α·β, we have:

$$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \ \Rightarrow \ \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = S^2 - 2P$$

$$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 \ \Rightarrow \ \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = S^3 - 3PS$$

$$\alpha^3 + \alpha^2 + \alpha + \beta^3 + \beta^2 + \beta = S^3 - 3PS + S^2 - 2P + S = 4^3 - 3\cdot 4\cdot 22 - 2\cdot 22 + 4 = -224$$

October 12-13: Let the polynomial be given:

$$\mathbf{P(x) = x^3 + x - m}$$

and a, b and c its roots. Find m for that

$$\mathbf{\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b} = m} \quad (*)$$

Solution: First, let us note that (*) implies that a, b and c are non-zero (if any root is, it would not make sense to divide by it). And by the last formula of Vietà we will have m ≠ 0 (since m = abc)

We have:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b} = m \ \Leftrightarrow \ abc\left(\frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2}\right) = m \ \Leftrightarrow \ \left(\frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2}\right) = 1 \ (**)$$

Let's consider the polynomial in t: Q (t) = P (1/t). Let's find the roots of Q (t) = 0. Being P (x) = (x - a) · (x - b) · (x - c), we have:

$$Q(t) = \left(\frac{1}{t} - a\right)\cdot\left(\frac{1}{t} - b\right)\cdot\left(\frac{1}{t} - c\right) = 0 \ \Rightarrow \ t \in \left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$$

But:

$$0 = Q(t) = P\left(\frac{1}{t}\right) = \frac{1}{t^3} + \frac{1}{t} - m = \frac{-mt^3 + t^2 + 1}{t^3} \ \Leftrightarrow \ -mt^3 + t^2 + 1 = 0$$

And, applying the Vietà formulas to this polynomial, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{m}$$

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}=0$$

$$\frac{1}{abc}=\frac{1}{m}$$

At last, given (**):

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=\begin{cases}=\frac{1}{m^2}\\=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)\end{cases}$$

$$\frac{1}{m^2}=1+2\cdot 0=1 \quad\Rightarrow\quad m=\pm 1$$

<u>October14:</u> If α and β are the solutions of x^2 - 9x - 70 = 0, calculate the value of $|\alpha-\beta|$

<u>Solution:</u> We are asked to calculate:

$$|\alpha-\beta|=+\sqrt{(\alpha-\beta)^2}=+\sqrt{\alpha^2+\beta^2-2\alpha\beta} \qquad (*)$$

By the Vietà formulas, we have:

$$S=\alpha+\beta=9; \quad P=\alpha\cdot\beta=-70$$

And how:

$$(\alpha+\beta)^2=\begin{cases}=9^2\\=\alpha^2+\beta^2+2\alpha\beta\end{cases}$$

We have:

$$\alpha^2+\beta^2=9^2-2\cdot(-70)=221$$

Finally, in (*):

$$|\alpha-\beta|=+\sqrt{221-2\cdot(-70)}=19$$

<u>October 15:</u> If r and s are the solutions of equation:

$$\mathbf{0 = x^2 + 2020x + 2019}$$

calculate the value of

r·(1 - r) + s·(1 - s)

<u>Solution:</u> WE have:

$$r(1-r)+s(1-s)=-(r^2+s^2)+r+s \quad (*)$$

From the Vietà formulas:

$$S=r+s=-2020; \quad rs=2019$$

And, since:

$$(r+s)^2=\begin{cases}=(-2020)^2\\=r^2+s^2+2rs\end{cases} \qquad \Rightarrow r^2+s^2=(-2020)^2-2\cdot 2019=4076362$$

For last, in (*)

$$r(1-r)+s(1-s)=-(r^2+s^2)+r+s=-4076362-2020=-4078382$$

October 16-17: Solve the system:

$$\left.\begin{array}{r} \mathbf{x+\ y+\ z+\ t\ =\ 2} \\ \mathbf{xy+\ xz+\ xt+\ yz+zt+yt\ =-7} \\ \mathbf{xyz+\ xyt+\ xzt+\ yzt\ =\ -8} \\ \mathbf{xyzt\ =\ 12} \end{array}\right\}$$

Solution: Solving the above system is equivalent to solving

$$u^4+bu^3+cu^2+du+e=0$$

fulfilling the coefficients

sum of solutions of the equation = - b = 2

sum of products of two solutions of the equation = c = - 7

sum of products of three solutions of the equation = - d = - 8

product of the four solutions of the equation = e = 12.

The resulting equation is:

$$u^4-2u^3-7u^2+8u+12=0$$

Solve for Ruffini:

	1	-2	-7	8	12
-1		-1	3	4	-12
	1	-3	-4	12	0
-2		-2	10	-12	
	1	-5	6	0	
2		2	-6		
	1	-3	0		

With that:

$$u^4-2u^3-7u^2+8u+12=0=(u+1)\cdot(u+2)\cdot(u-2)\cdot(u-3)\ \Rightarrow\ \begin{cases} u=-1 \\ u=-2 \\ u\ =\ 2 \\ u\ =\ 3 \end{cases}$$

That is, the solutions of the system are (x, y, z, t) = (-1, -2, 2, 3) and the permutations of the numerical vector

October 19: Let α and β the roots of the equation $x^2 - 8x + 9 = 0$.

Calculate:

$$\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2$$

Solution: We have:

$$\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2=\alpha^2+\frac{1}{\alpha^2}-2+\beta^2+\frac{1}{\beta^2}-2=(\alpha^2+\beta^2)+\frac{1}{\alpha^2}+\frac{1}{\beta^2}-4$$
$$=(\alpha^2+\beta^2)+\frac{(\alpha^2+\beta^2)}{\alpha^2\cdot\beta^2}-4=\{\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta\}$$
$$=(\alpha+\beta)^2-2\alpha\beta+\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}-4$$

From the Vietà formulas:

$$\alpha+\beta=8;\quad \alpha\beta=9$$

With what:

$$\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\beta-\frac{1}{\beta}\right)^2=(\alpha+\beta)^2-2\alpha\beta+\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}-4=8^2-2\cdot 9+\frac{8^2-2\cdot 9}{9^2}-4=\frac{3448}{81}$$

October 20: Let s and r the solutions of the equation:

$$x^2 - 2020x + a^2 - 4040a + 4080400 = 0$$

Calculate a so that s·r is minimal

Solution: From the Vietà formulas:

$$s\cdot r=a^2-4040a+4080400=(a-2020)^2$$

Therefore, s·r will be minimal when a = 2020.

October 21: Calculate $a^2 + b^2 + c^2$ where a, b and c are the solutions of the equation:

$$\mathbf{3x^3-2x^2+5x-7=0}$$

Solution: From the Vietà formulas we have:

$$a+b+c=\frac{2}{3};\quad ab+ac+bc=\frac{5}{3};\quad abc=\frac{7}{3}$$

But:

$$(a+b+c)^2=\left\{\begin{matrix} =\left(\frac{2}{3}\right)^2=\frac{4}{9} \\ =a^2+b^2+c^2+2(ab+bc+ac)=a^2+b^2+c^2+2\cdot\frac{5}{3}\end{matrix}\right\}$$
$$\Rightarrow a^2+b^2+c^2=-\frac{26}{9}$$

Therefore, the polynomial has two opposite complex roots and a real root.

October 22-23: Let a, b and c the roots of the equation:

$$\mathbf{0 = 2x^3 - x^2 + 3x - 1}$$

Find the equation with roots

$$\boldsymbol{\alpha=\frac{a+1}{a+2(b+c)};\ \beta=\frac{b+1}{b+2(a+c)};\ \eta=\frac{c+1}{c+2(a+b)}}$$

Solution: We have:

$$\alpha = \frac{a+1}{a+2(b+c)} = \frac{a+1}{2(a+b+c)-a} = \frac{a+1}{2\frac{1}{2}-a} = \frac{a+1}{1-a}$$

$$\beta = \frac{b+1}{b+2(a+c)} = \frac{b+1}{2(a+b+c)-b} = \frac{b+1}{2\frac{1}{2}-b} = \frac{b+1}{1-b}$$

$$\eta = \frac{c+1}{c+2(a+b)} = \frac{c+1}{2(a+b+c)-c} = \frac{c+1}{2\frac{1}{2}-c} = \frac{c+1}{1-c}$$

Therefore, the relationship between the roots of the given equation and the one requested is:

$$y = \frac{x+1}{1-x}$$

Solving for x, we have:

$$y(1-x) = 1+x \;\; \Rightarrow \;\; y-1 = x+yx = x(1+y) \;\Rightarrow x = \frac{y-1}{y+1}$$

And substituting in the original equation:

$$0 = 2\left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right)^2 + 3\left(\frac{y-1}{y+1}\right) - 1 \Rightarrow \cdots \Rightarrow 3y^3 - 5y^2 + y - 7 = 0$$

October 24: Obtain Vietà's formulas for third degree polynomials

Solution: We have, if x_1, x_2 and x_3 are the roots of $ax^3 + bx^2 + cx + d = 0$:

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(x-x_1)\cdot(x-x_2)\cdot(x-x_3) = a\cdot(x^2 - x_1x - x_2x + x_1x_2)\cdot(x-x_3) \\ &= a(x^3 - x_1x^2 - x_2x^2 + x_1x_2x - x_3x^2 + x_1x_3x + x_2x_3x - x_1x_2x_3) \\ &= ax^3 + a(-x_1 - x_2 - x_3)x^2 + a(x_1x_2 + x_1x_3 + x_3x_2)x - a(x_1x_2x_3) \end{aligned}$$

Equating coefficients:

$$b = -a(x_1 + x_2 + x_3) \quad \Rightarrow \quad x_1 + x_2 + x_3 = -\frac{b}{a}$$

$$c = a(x_1x_2 + x_1x_3 + x_3x_2) \;\Rightarrow\; x_1x_2 + x_1x_3 + x_3x_2 = \frac{c}{a}$$

$$d = -a(x_1x_2x_3) \quad \Rightarrow \quad x_1x_2x_3 = -\frac{d}{a}$$

October 26: Obtain Vietà's formulas for a polynomial of degree four

Solution: We have, if x_1, x_2, x_3 and x_4 are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$:

$$\begin{aligned} ax^4 + bx^3 + cx^2 + dx + e &= a(x-x_1)\cdot(x-x_2)\cdot(x-x_3)\cdot(x-x_4) \\ &= a\cdot(x^2 - x_1x - x_2x + x_1x_2)\cdot(x^2 - x_3x - x_4x + x_3x_4)\cdot \\ &= a(x^4 - x_1x^3 - x_2x^3 + x_1x_2x^2 - x_3x^3 + x_1x_3x^2 + x_2x_3x^2 - x_1x_2x_3x - x_4x^3 \\ &+ x_1x_4x^2 + x_2x_4x^2 - x_1x_2x_4x + x_3x_4x^2 - x_1x_3x_4x - x_2x_3x_4x + x_1x_2x_3x_4) \\ &= ax^4 + a(-x_1 - x_2 - x_3 - x_4)x^3 + a(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2 \\ &- a(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) + a(x_1x_2x_3x_4) \end{aligned}$$

Equating coefficients:

$$b = -a(x_1 + x_2 + x_3 + x_4) \quad \Rightarrow \quad x_1 + x_2 + x_3 + x_4 = -\frac{b}{a}$$

$$c = a(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \Rightarrow x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{c}{a}$$

$$d = -a(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \Rightarrow x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -\frac{d}{a}$$

$$e = a(x_1x_2x_3x_4) \quad \Rightarrow \quad x_1x_2x_3x_4 = \frac{e}{a}$$

October 27-28: Let $f(x) = (x^2 + 10x + 25)^{1010} - 3x + 2$ and r_i for $i \in \{1, 2, \dots, 2020\}$ its roots. Calculate:

$$\sum_{i=1}^{2020} (r_i + 5)^{2020}$$

Solution: We have:

$$0 = f(r_i) = (r_i^2 + 10r_i + 25)^{1010} - 3r_i + 2 = (r_1 + 5)^{2020} - 3r_i + 2 \Rightarrow (r_1 + 5)^{2020} = 3r_i - 2$$

Therefore:

$$\sum_{i=1}^{2020} (r_i + 5)^{2020} = 3\sum_{i=1}^{2020} r_i - 2 \cdot 2020 = 3S - 4040$$

To calculate S we develop f (x) and apply the Vietà formulas:

$$f(x) = (x+5)^{2020} - 3x + 2 = x^{2020} + \binom{2020}{1}x^{2019} \cdot 5 + \cdots \quad \Rightarrow \quad S = -5 \cdot \binom{2020}{1} = -10100$$

Finally:

$$\sum_{i=1}^{2020} (r_i + 5)^{2020} = 3S - 4040 = 3 \cdot (-10100) - 4040 = -34340$$

October 29: Find three numbers whose sum is 6, the sum of their squares 38, and the sum of their cubes 144

Solution: It asks to solve the system:

$$\left.\begin{aligned} a + b + c = 6 \\ a^2 + b^2 + c^2 = 38 \\ a^3 + b^3 + c^3 = 144 \end{aligned}\right\}$$

But:

$$(a+b+c)^2 = \begin{cases} = 6^2 = 36 \\ = a^2 + b^2 + c^2 + 2(ab + ac + bc) = 38 + 2(ab + ac + bc) \end{cases}$$

Then if a + b + c = 6:

$$a^2 + b^2 + c^2 = 38 \quad \Leftrightarrow \quad ab + ac + bc = -1$$

But:

$$(a+b+c)^3 = \begin{cases} = 6^3 = 216 \\ = a^3 + b^3 + c^3 + 3ac^2 + 3bc^2 + 3cb^2 + 3ab^2 + 3ca^2 + 3ba^2 + 6abc \end{cases}$$

$$(a+b+c)^3 = \begin{cases} = 6^3 = 216 \\ = a^3+b^3+c^3+3c^2(a+b)+3b^2(c+a)+3a^2(c+b)+6abc \end{cases}$$

$$(a+b+c)^3 = \begin{cases} = 6^3 = 216 \\ = a^3+b^3+c^3+3c^2(S-c)+3b^2(S-b)+3a^2(S-a)+6abc \end{cases}$$

$$(a+b+c)^3 = \begin{cases} = 6^3 = 216 \\ = a^3+b^3+c^3-3c^3+3c^2S-3b^3+3b^2S-3a^3+3a^2S+6abc \end{cases}$$

$$(a+b+c)^3 = \begin{cases} = 6^3 = 216 \\ = a^3+b^3+c^3-3c^3-3b^3-3a^3+3S(c^2+b^2+a^2)+6abc \end{cases}$$

$$(a+b+c)^3 = \begin{cases} = 6^3 = 216 \\ = -2(c^3+b^3+a^3)+3S(c^2+b^2+a^2)+6abc \end{cases}$$

Therefore, if a + b + c = 6 and if $a^2 + b^2 + c^2 = 38$:

$$a^3+b^3+c^3 = 144 \quad \Leftrightarrow \quad abc = -30$$

In short, by the formulas of Vietà:

$$\left.\begin{matrix} a+b+c=6 \\ a^2+b^2+c^2=38 \\ a^3+b^3+c^3=144 \end{matrix}\right\} \Leftrightarrow \left.\begin{matrix} a+b+c=6 \\ ab+ac+bc=-1 \\ abc=-30 \end{matrix}\right\} \Leftrightarrow x^3-6x^2-x+30=0$$

And solving for Ruffini:

	1	-6	-1	30
-2		-2	16	-30
	1	-8	15	0
3		3	-15	
	1	-5	0	

$$0 = x^3-6x^2-x+30 = (x+2)(x-3)(x-5)$$

Then the numbers searched are -2, 3 and 5 (in any order)

October 30: Let the polynomial be:

$$\mathbf{P(x) = x^3 - mx^2 + 3mx - m}$$

and a, b and c its roots. Find m for that $\mathbf{a^3 + b^3 + c^3 > -5}$

Solution: We have:

$$\begin{aligned}(a+b+c)^3 &= a^3+b^3+c^3+3ac^2+3bc^2+3cb^2+3ab^2+3ca^2+3ba^2+6abc \\ &= a^3+b^3+c^3+3c^2(a+b)+3b^2(c+a)+3a^2(c+b)+6abc \\ &= a^3+b^3+c^3+3c^2(S-c)+3b^2(S-b)+3a^2(S-a)+6abc = a^3+b^3+c^3 \\ &-3c^3+3c^2S-3b^3+3b^2S-3a^3+3a^2S+6abc = a^3+b^3+c^3-3c^3-3b^3-3a^3 \\ &+3(a+b+c)(c^2+b^2+a^2)+6abc \\ &= -2(c^3+b^3+a^3)+3(a+b+c)(c^2+b^2+a^2)+6abc\end{aligned}$$

From where:

$$2(c^3+b^3+a^3) = -(a+b+c)^3+3(a+b+c)(c^2+b^2+a^2)+6abc$$

But:

$$(a^2+b^2+c^2) = (a+b+c)^2-2(ab+ac+bc)$$

With what:

$$2(c^3+b^3+a^3) = -(a+b+c)^3 + 3(a+b+c)\big((a+b+c)^2 - 2(ab+ac+bc)\big) + 6abc =$$
$$= -(a+b+c)^3 + 3(a+b+c)^3 - 6(a+b+c)(ab+ac+bc) + 6abc$$
$$= 2(a+b+c)^3 - 6(a+b+c)(ab+ac+bc) + 6abc$$

And therefore:

$$(c^3+b^3+a^3) = (a+b+c)^3 - 3(a+b+c)(ab+ac+bc) + 3abc$$

From the Vietà formulas:

$$a+b+c = m;\;\; ab+ac+bc = 3m;\;\; abc = m$$

With that:

$$-5 < \; a^3+b^3+c^3 = m^3 - 3m\cdot 3m + 3m; \quad m^3 - 9m^2 + 3m + 5 > 0$$

$$(m-1)(m^2-8m+5) > 0; \quad m \in \,]4-\sqrt{21}; 1[\, \cup \,]4+\sqrt{21};\, +\infty[$$

<u>October 31:</u> Solve the equation:

$$\mathbf{(ax - b)^2 + (bx - a)^2 = x}$$

knowing that it has an integer root and that a, b$\in\mathbb{Z}$

<u>Solution:</u> We have:

<u>Motto:</u> In the context of the statement:

a = b = 0 sii x = 0

$\Rightarrow$ We have: $$(0\cdot x - 0)^2 + (0\cdot x - 0)^2 = \begin{cases} = x \\ = 0 \end{cases}$$

$\Leftarrow$ We have: $0 = x = (a\cdot 0 - b)^2 + (b\cdot 0 - a)^2 = (-b)^2 + (-a)^2 = b^2 + a^2 \Rightarrow \begin{cases} b^2 = 0 \Rightarrow b = 0 \\ a^2 = 0 \Rightarrow a = 0 \end{cases}$

<u>Note:</u> Note that if a $\neq$ 0 or b $\neq$ 0 and x_1 is the integer root of the equation in the statement, then $x_1 > 0$, since:

$$\left.\begin{matrix} (ax_1 - b)^2 \geq 0 \\ (bx_1 - a)^2 \geq 0 \end{matrix}\right\} \Rightarrow x_1 = (ax_1 - b)^2 + (bx_1 - a)^2 \geq 0$$

But if $x_1 = 0$ then, by the motto a = b = 0 against that a $\neq$ 0 o b $\neq$ 0.

<u>Demonstration:</u> Let the equation:

$$(ax - b)^2 + (bx - a)^2 = x \quad (*)$$

with a or b or both not null, then developing and grouping, we have:

$$(a^2+b^2)x^2 - (1+4ab)x + a^2 + b^2 = 0$$

Let x_1 ($\in\mathbb{Z}$) and x_2 the roots of the equation. Since both are real, the discriminant of the equation is non-negative. Therefore:

$$0 \leq (4ab+1)^2 - 4(a^2+b^2)^2 = (2a^2+2b^2+4ab+1)(4ab+1-2a^2-2b^2)$$
$$= (2(a+b)^2+1)(1-2(a-b)^2)$$

And how $2(a+b)^2 + 1 \geq 0$, we have:

$$1 - 2(a-b)^2 \geq 0 \;\; \Rightarrow \;\; 1 \geq 2(a-b)^2 \geq 0 \; (a, b \in \mathbb{Z}) \; \Rightarrow (a-b)^2 = 0 \Rightarrow a = b$$

with which the equation (*) remains:

$$2a^2x^2 - (4a^2+1)x + 2a^2 = 0$$

By the formulas of Vietà:

$$x_1 + x_2 = \frac{4a^1+1}{2a^2} = 2 + \frac{1}{2a^2}$$
$$x_1 \cdot x_2 = 1 \ \text{(nota)} \ x_1 \text{ y } x_2 \text{ son ambas positivas}$$

Then:

$$x_1 < x_1 + x_2 = 2 + \frac{1}{2a^2} < 3 \ \left(\text{pues } \frac{1}{2a^2} < 1 \text{ ya que } a \in \mathbb{Z}\right)$$

What's more $x_1 \neq 1$, since if $x_1 = 1$

$$0 = 2a^2 - (4a^2+1) + 2a^2 = -1 \quad \text{absurd!}$$

Then:

$$2 \leq x_1 < 3 \quad \Rightarrow \quad x_1 = 2 \quad \Rightarrow \quad x_2 = \frac{1}{2} \ \ (\text{ya que } x_1 \cdot x_2 = 1)$$

Then the solutions of the equation are:

$$a = b = 0 \quad \Rightarrow \quad x = 0$$
$$a \neq 0 \text{ o } b \neq 0 \ \Rightarrow \ x_1 = 2, \ x_2 = \frac{1}{2}$$