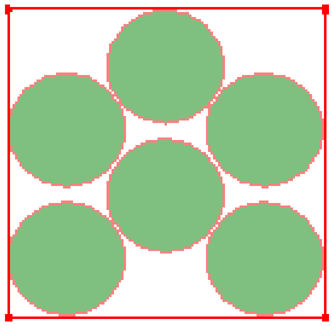
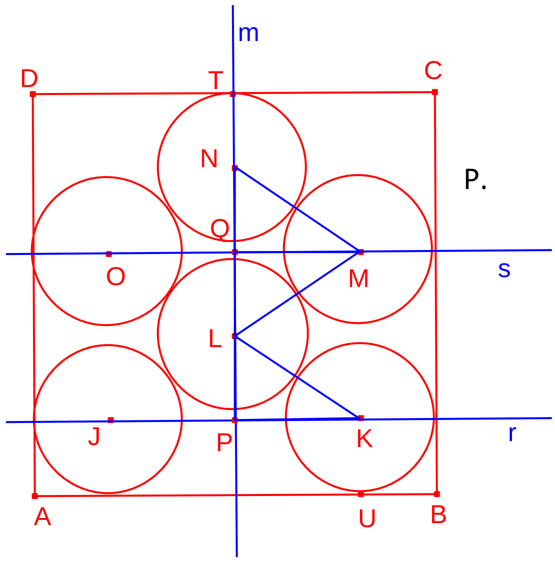
DECEMBER 2020 SOLUTIONS

PROBLEMS FOR USING GEOMETRIC PROGRAMS. AUTHOR: RICARD PEIRÓ I ESTRUCH. IES “Abastos”, València

December 1-2: To pack 6 equal circles in a square, the distribution of the figure has to be done (demonstrated by Graham in 1963). Determine the ratio between the side of the square and the radius of the circles



Solution: Let the square ABCD be of side $\overline{AB} = c$



Let J, K, L, M, N, O the centres of the six circles of radius r. Let us consider the line m, bisector of the side $\overline{AB}$. Let r be the line that passes through the centers J, K. Let s be the line that passes through the centres O and M. The lines m and r intersects at point P. The lines m and s intersect at point Q.

$$\overline{JK} = c - 2r.$$

$$\overline{PQ} = \overline{QM} = \frac{c-2r}{2}, \overline{LK} = \overline{MN} = \overline{LM} = 2r$$

So, the right triangles $\overset{\Delta}{LPK}, \overset{\Delta}{LQM}, \overset{\Delta}{NQM}$ are equal. So

$$\overline{LP} = \overline{QL} = \overline{QN}$$

$$2r + 3\overline{LQ} = c$$

$$\overline{LQ} = \frac{c-2r}{3}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{LPK}$

$$(2r)^2 = \left(\frac{c-2r}{2}\right)^2 + \left(\frac{c-2r}{3}\right)^2$$

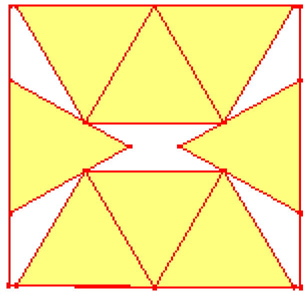
Simplify:

$$92r^2 + 52cr - 13c^2 = 0$$

$$92\left(\frac{r}{c}\right)^2 + 52\left(\frac{r}{c}\right) - 13 = 0$$

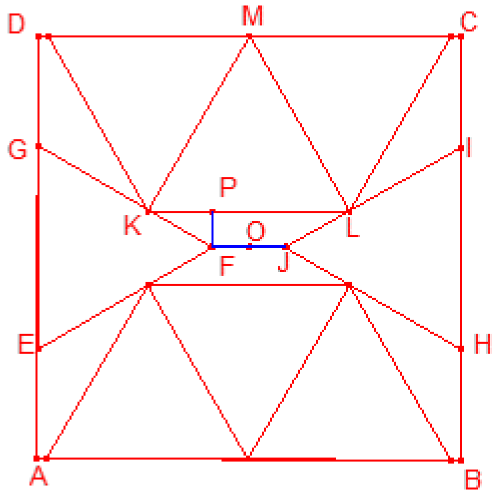
Solving the equation:

$$\frac{r}{c} = \frac{-13 + 6\sqrt{13}}{46}$$



December 3-4: To pack eight equal equilateral triangles into a square, they must be placed as in the figure (proved by Erich Friedman in 1966). Determine the ratio of the side of the square and the side of the triangle

Solution: Let the square ABCD be of side $\overline{AB} = c$ and the centre O. Let the triangles be equilateral $\overset{\Delta}{EFG}, \overset{\Delta}{HIJ}, \overset{\Delta}{KLM}$ of side $\overline{GE} = \overline{KL} = x$. Let P be the projection of F on $\overline{KL}$



$$\angle PKF = 30°$$

$$\overline{FO} = \frac{c - x\sqrt{3}}{2}, \overline{PF} = \frac{c - x\sqrt{3}}{2}$$

$$\overline{KP} = \frac{\overline{KL}}{2} - \overline{FO} = \frac{x - c + x\sqrt{3}}{2} = \frac{(1+\sqrt{3})x - c}{2}$$

Applying trigonometric ratios to the right triangle $\overset{\Delta}{KPF}$

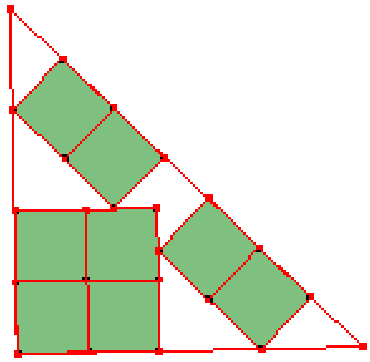
$$\frac{c - x\sqrt{3}}{(1+\sqrt{3})x - c} = \frac{\sqrt{3}}{3}$$

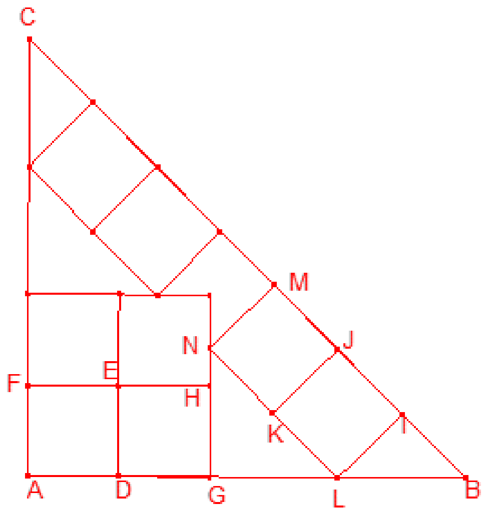
$$(3+\sqrt{3})c = (4\sqrt{3}+3)x$$

The ratio between the side of the square and the side of the equilateral triangle is:

$$\frac{c}{x} = \frac{4\sqrt{3}+3}{3+\sqrt{3}} = \frac{3\sqrt{3}-1}{2}$$

December 5-12: To pack eight equal squares into an isosceles right triangle, they must be placed as in the figure (tested by Erich Friedman in 2005). Determine the ratio between the leg of the triangle and the side of the square

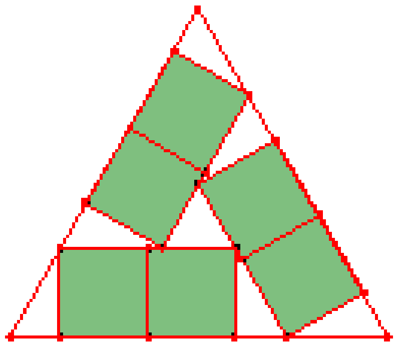




Solution: Let the right triangle be $\overset{\Delta}{ABC}$, with $A = 90°$ and the side $\overline{AB} = c$. Let the squares ADEF, DGHE, IJKL, JMNK of sides $\overline{AD} = \overline{DG} = \overline{IJ} = \overline{JM} = x$. Then:

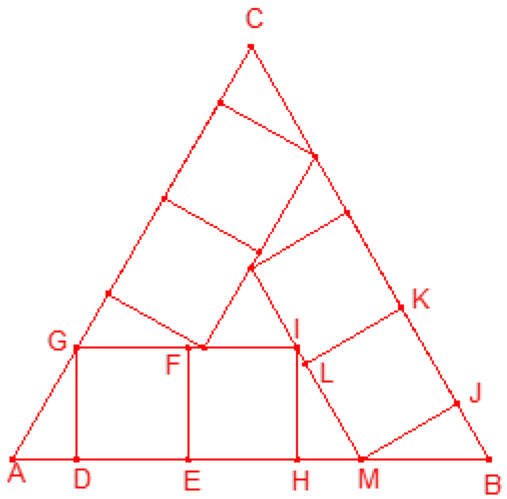
$$\overline{GL} = \overline{LB} = x\sqrt{2}$$
$$\overline{AB} = c = \left(2 + 2\sqrt{2}\right)x$$
$$\frac{c}{x} = 2 + 2\sqrt{2}$$

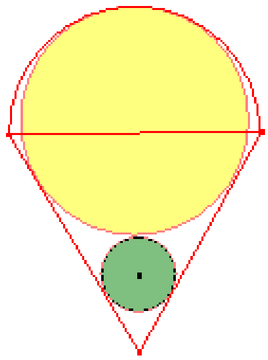
December 7-8: To pack 6 equal squares in an equilateral triangle, the distribution of the figure has to be done (demonstrated by Erich Friedman in 1997). Determine the ratio between the side of the triangle and the side of the square



Solution: Let the equilateral triangle $\overset{\Delta}{ABC}$ with side $\overline{AB} = c$. Let the squares DEFG, EHIF, JKLM of side $\overline{DE} = x, \overline{EH} = x, \overline{JK} = x$

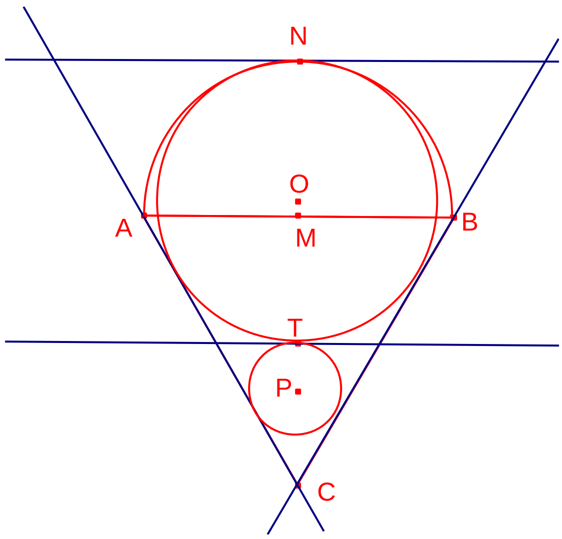
$$\overline{AD} = \overline{HM} = \frac{\sqrt{3}}{3}x, \overline{MB} = \frac{2\sqrt{3}}{3}$$
$$\overline{AB} = c = \left(2 + \frac{4\sqrt{3}}{3}\right)x$$
$$\frac{c}{x} = 2 + \frac{4\sqrt{3}}{3}$$





December 9-16: On one side of an equilateral triangle a semicircle has been drawn. A circle is interior tangent to the semicircle and to two sides of the triangle. Another circle is tangent to the outside of the previous circle and to the same sides. Determine the ratio between the radii of the two circles

Solution: Let $\overset{\Delta}{ABC}$ the equilateral triangle with side $\overline{AB} = c$. Let M be the midpoint of side $\overline{AB}$. Let N be the midpoint of the semicircle and point of tangency. Let O be the centre of the large circle of radius $\overline{ON} = r$. Applying the Pythagorean theorem to the triangle $\overset{\Delta}{AMC}$:



$$\overline{MC} = \frac{\sqrt{3}}{2}c$$

$$\overline{NC} = \overline{MC} + \frac{c}{2} = \frac{1+\sqrt{3}}{2}c$$

$$\overline{ON} = \frac{1}{3}\overline{NC}$$

$$r = \frac{1+\sqrt{3}}{6}c$$

Let T be the point of tangency of the two circles. Let P be the centre of the small circle of radius $\overline{PT} = s$

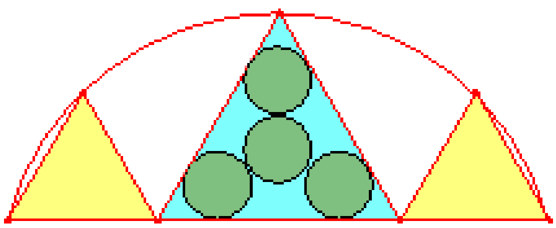
$$\overline{TC} = \overline{NC} - 2r = \frac{1+\sqrt{3}}{6}c$$

$$\overline{PT} = \frac{1}{3}\overline{TC}$$

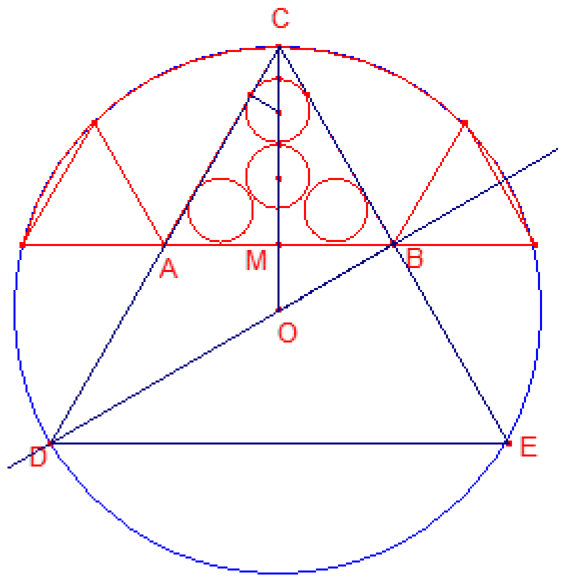
$$s = \frac{1+\sqrt{3}}{18}c$$

$$\frac{s}{r} = \frac{1}{3}$$

December 10-11: Three equilateral triangles have been drawn on a chord of circumference of radius R. In the central one, four equal circles of radius r have been inscribed. Find the relationship between r and R



Solution: Let us consider the circumference of centre O and radius R. Let $\overset{\Delta}{ABC}$ be the central equilateral triangle with side $\overline{AB} = a$. Let M be the midpoint of side $\overline{AB}$. Note that $\overline{CM} = 6r$. We draw the equilateral triangle inscribed in the circumference of radius R. Note that OB is perpendicular to CE since OB is the bisector of the equilateral triangle on the right.



$$\overline{OB} = \frac{R}{2}$$

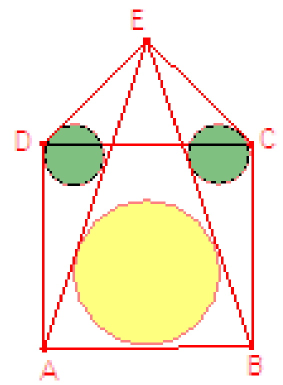
$$\overline{OM} = \frac{1}{2}\overline{OB} = \frac{R}{4}$$

$$6r + \frac{R}{4} = R$$

Then, $r = \frac{1}{8}R$

December 14-15: An isosceles right triangle ΔCDE has been drawn on the side AB of a square ABCD, where∠CED = 90°.

Calculate the ratio between the radii of the circles inscribed in the triangles ΔADE and ΔABE.



Solution: Let $\overline{AB} = c$ the side of the square ABCD. Let M and N be the midpoints of the sides $\overline{AB}$ and $\overline{CD}$, respectively. Let O be the centre of the circle inscribed to the triangle $\overset{\Delta}{ABE}$ of radius $r = \overline{OM}$.

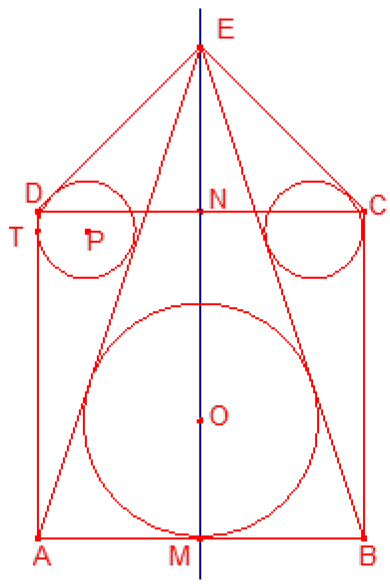
$$\overline{NE} = \overline{CN} = \frac{c}{2}, \overline{AM} = \frac{c}{2}$$

$$\overline{ME} = \frac{3c}{2}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{AME}$:

$$\overline{AE} = \frac{\sqrt{10}}{2}$$

The area of the triangle $\overset{\Delta}{ABE}$ is:



$$S_{ABE} = \frac{1}{2}c\frac{3}{2}c = \frac{1 + 2\frac{\sqrt{10}}{2}}{2}cr$$

Isolating r:

$$r = \frac{\sqrt{10} - 1}{6}c$$

Let P be the centre of the circle inscribed to the triangle $\overset{\Delta}{ADE}$ of radius $s = \overline{PT}$. The area of the triangle $\overset{\Delta}{ADE}$ is:

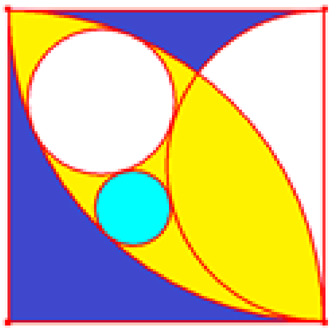
$$S_{ADE} = \frac{1}{2}c\frac{1}{2}c = \frac{1 + \frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2}}{2}cs$$

Isolating s:

$$s = \frac{1}{2 + \sqrt{2} + \sqrt{10}}c$$

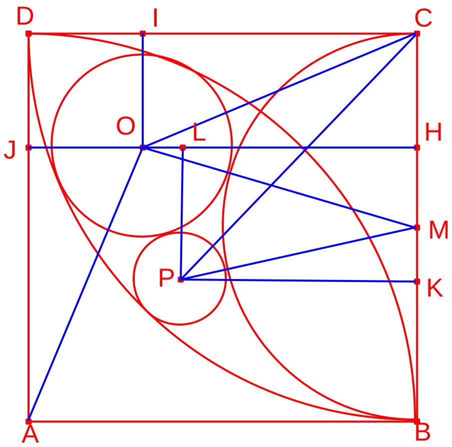
The ratio between the radii is

$$\frac{s}{r}=\frac{\frac{1}{2+\sqrt{2}+\sqrt{10}}}{\frac{\sqrt{10}-1}{6}}=\frac{6}{8-\sqrt{2}+2\sqrt{5}+\sqrt{10}}$$



December 17-24: Inside a square, two quadrants with centres, two opposite vertices and a semicircle of diameter on one side, have been drawn. A tangent circle has been drawn inside the quadrants and outside the semicircle. Another circle is external tangent to the previous one, internal tangent to a quadrant and external tangent to the semicircle. Find the ratio of the radii of the two circles

Solution: Let ABCD be the square with side $\overline{AB}=1$. Let O be the centre of the circle tangent inside the quadrants and outside the semicircle, of radius r. Let H be the projection of O onto the side $\overline{AB}$. Let $\overline{CH}=\overline{OI}=\overline{OJ}=x$.



$$\overline{OH}=1-x,\overline{OC}=1-r,\overline{OM}=\frac{1}{2}+r,\overline{MH}=\frac{1}{2}-x$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OHC}$:

$$(1-r)^2=x^2+(1-x)^2$$

Simplifying:

$$r^2-2r=2x^2-2x \qquad (1)$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OHM}$:

$$\left(\frac{1}{2}+r\right)^2=\left(\frac{1}{2}-x\right)^2+(1-x)^2$$

Simplifying:

$$r^2+r=2x^2-3x+1 \qquad (2)$$

Subtracting the expressions (1) and (2)

$$x=1-3r \qquad (3)$$

Substituting expression (3) in expression (1)

$$r^2-2r=2(1-3r)^2-2(1-3r)$$

Solving the equation:

$$r=\frac{4}{17}$$

Then, $x=\frac{5}{17},\overline{OH}=\frac{12}{17}$

Let P be the centre of the circle tangent outside to the circle with centre O, inside tangent to a quadrant, and outside tangent to the semicircle. Let s be your radius.

Let K be the projection of P onto the side $\overline{\mathrm{AB}}$

Let L be the projection of P onto $\overline{\mathrm{OH}}$.

Let $\overline{\mathrm{OL}} = \mathrm{y}$, and $\overline{\mathrm{MK}} = \mathrm{z}$

$$\overline{\mathrm{OP}} = \frac{4}{17} + \mathrm{s}, \overline{\mathrm{PL}} = \overline{\mathrm{HK}} = \frac{7}{34} + \mathrm{z}$$

$$\overline{\mathrm{PM}} = \frac{1}{2} + \mathrm{s}, \overline{\mathrm{PK}} = \frac{12}{17} - \mathrm{y}$$

$$\overline{\mathrm{PC}} = 1 - \mathrm{s}, \overline{\mathrm{CK}} = \frac{1}{2} + \mathrm{z}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{PKM}}$:

$$\left(\frac{1}{2} + \mathrm{s}\right)^2 = \mathrm{z}^2 + \left(\frac{12}{17} - \mathrm{y}\right)^2 \qquad (4)$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{PKC}}$:

$$(1 - \mathrm{s})^2 = \left(\frac{1}{2} + \mathrm{z}\right)^2 + \left(\frac{12}{17} - \mathrm{y}\right)^2 \qquad (5)$$

Subtracting expressions (4) and (5) and simplifying:

$$\mathrm{z} = \frac{1}{2} - 3\mathrm{s} \qquad (6)$$

$$\overline{\mathrm{OP}} = \frac{4}{17} + \mathrm{s}, \overline{\mathrm{PL}} = \overline{\mathrm{HK}} = \frac{7}{34} + \mathrm{z} = \frac{12}{17} - 3\mathrm{s}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{OLP}}$:

$$\left(\frac{4}{17} + \mathrm{s}\right)^2 = \mathrm{y}^2 + \left(\frac{12}{17} - 3\mathrm{s}\right) \qquad (7)$$

Simplifying:

$$-8\mathrm{s}^2 + \frac{80}{17}\mathrm{s} = \mathrm{y}^2 + \frac{128}{289} \qquad (8)$$

Simplifying the expression (4)

$$\left(\frac{1}{2} + \mathrm{s}\right)^2 = \left(\frac{1}{2} - 3\mathrm{s}\right)^2 + \left(\frac{12}{17} - \mathrm{y}\right) \qquad (9)$$

$$-8\mathrm{s}^2 + 4\mathrm{s} = \mathrm{y}^2 - \frac{24}{17}\mathrm{y} + \frac{144}{289} \qquad (10)$$

Subtracting the expressions (9) and (10)

$$\mathrm{y} = \frac{2}{51} + \frac{1}{2}\mathrm{s} \quad (11)$$

Simplifying the expression (7)

$$\left(\frac{4}{17} + \mathrm{s}\right)^2 = \left(\frac{2}{51} + \frac{1}{2}\mathrm{s}\right)^2 + \left(\frac{12}{17} - 3\mathrm{s}\right) \qquad (12)$$

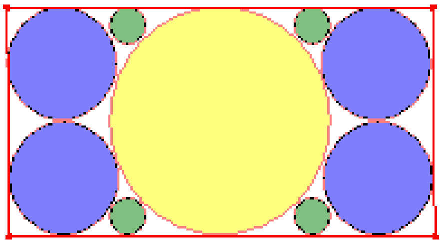
Solving the equation:

$$s = \frac{4}{33}$$

The ratio between the radii is:

$$\frac{s}{r} = \frac{17}{33}$$

December 18-19: In a rectangle, a central circumference tangent to the upper and lower sides has been drawn, with radius R. Four equal outer tangent circumferences have been added to the circumference and to the sides of the rectangle. The four small inner tangent circles have been added to one side of the rectangle and to the previous circles. Find the dimensions of the rectangle and the radii of the circles



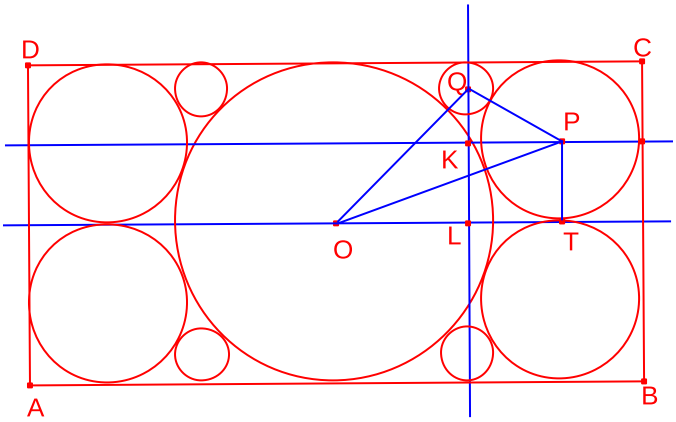
Solution: Let the rectangle ABCD. $\overline{AD} = 2R$. Let O be the center of the central circle of radius R. The radius of the four corner circles is $\frac{R}{2}$. Let P be the center of the circle tangent to two sides

$$\overline{OP} = R + \frac{R}{2} = \frac{3R}{2}, \overline{PT} = \frac{R}{2}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OTP}$

$$\overline{OT} = R\sqrt{2}$$

Side $\overline{AB}$ of the rectangle is:



$$\overline{AB} = 2\left(\overline{OT} + \frac{R}{2}\right) = (2\sqrt{2} + 1)R$$

Let Q be the centre of the circle tangent to the two circles and to one side. Let s be your radius. Let the line r through P be parallel to the line AB. Let the line s through Q be parallel to the line AD. Let K be the intersection of the lines r and s. Let L be the intersection of the lines OT and s.

$$\overline{OQ} = R + s, \overline{LQ} = R - s$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OLQ}$

$$\overline{OL} = 2\sqrt{Rs}$$

$$\overline{PK} = R\sqrt{2} - 2\sqrt{Rs}, \overline{QK} = \frac{R}{2} - s, \overline{PQ} = \frac{R}{2} + s$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{PKQ}$

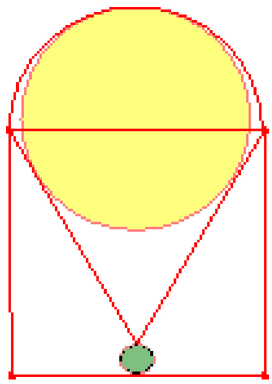
$$\left(\frac{R}{2} + s\right)^2 = \left(\frac{R}{2} - s\right)^2 + \left(R\sqrt{2} - 2\sqrt{Rs}\right)^2$$

Simply put:

$$s^2 - 6Rs + R^2 = 0$$

$$\left(\frac{s}{R}\right)^2 - 6\frac{s}{R} + 1 = 0$$

Solving the equation, $s = \left(3 - 2\sqrt{2}\right)R$.



<u>December 21-28:</u> On the side of a square we have drawn an inner equilateral triangle squared and an outer semicircle squared. A circle is tangent to the semicircle and to two sides of the triangle. Another circle passes through the vertex of the triangle and is tangent to one side of the square. Determine the ratio between radii.

<u>Solution:</u> Let ABCD be the side square $\overline{AB} = c$. Let ABC be the equilateral triangle $\overset{\Delta}{ABE}$. Let M be the midpoint of the side $\overline{AB}$. Let N be the midpoint of the semicircle and tangent point. Let O be the centre of the large circle of radius $\overline{ON} = r$.

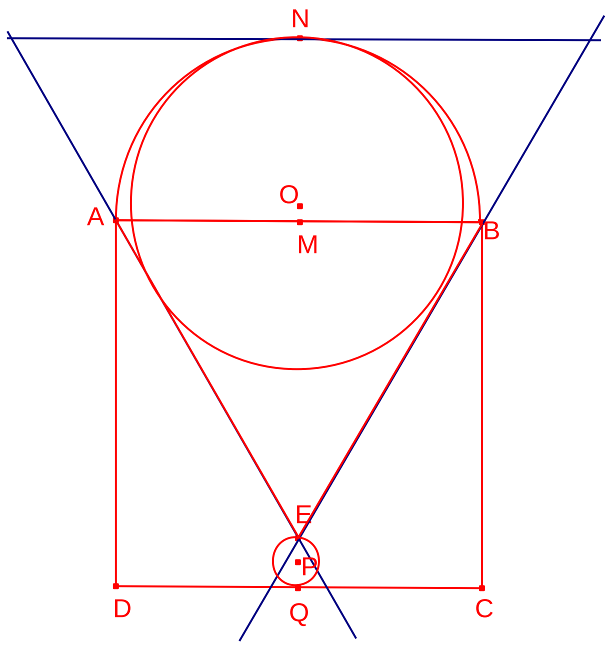
Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{AME}$:

$$\overline{ME} = \frac{\sqrt{3}}{2}c$$

$$\overline{NE} = \overline{ME} + \frac{c}{2} = \frac{1+\sqrt{3}}{2}c$$

$$\overline{ON} = \frac{1}{3}\overline{NE}$$

$$r = \frac{1+\sqrt{3}}{6}c$$



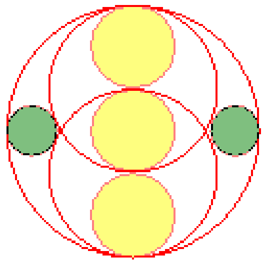
Let Q be the midpoint of the side $\overline{CD}$. Let P be the centre of the small circle of radius $s = \overline{PE} = \overline{PQ}$.

$$\overline{NQ} = \frac{3c}{2}$$

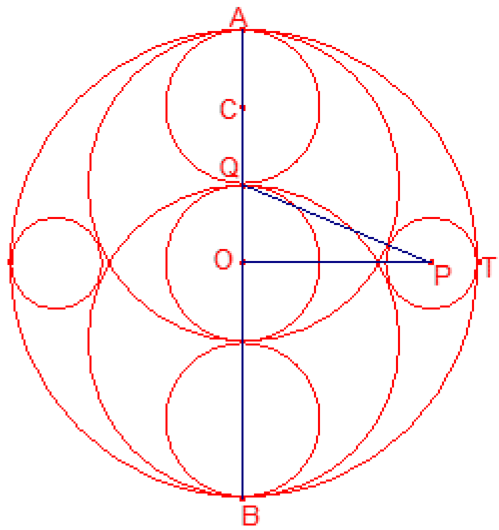
$$\overline{EQ} = \overline{NQ} - \overline{NE} = \frac{2-\sqrt{3}}{2}c$$

$$s = \frac{1}{2}\overline{EQ} = \frac{2-\sqrt{3}}{4}c$$

$$\frac{s}{r} = \frac{\frac{2-\sqrt{3}}{4}}{\frac{1+\sqrt{3}}{2}} = \frac{3\left(2\sqrt{3}-5\right)}{4}$$



December 22-23: Inside a circle of radius R, on a diameter, 3 circles of radius r_1 have been drawn. Two circles of radius r_2 have been drawn, interior tangents to that of radius R and exterior tangents to two of the three circles of radius r_1. Two circles of radius r_3 tangent interior to the radius R and tangents exterior to those of radius r_2 have been drawn. Calculate the ratio between r_3 and r_1



Solution: Consider the diameter $\overline{AB}$ of the outer circle of radius R. Let C be the center of the circle of radius $r_1 = \overline{CA}$.

$$6r_1 = 2R$$

Then, $r_1 = \frac{1}{3}R$. Let Q be the center of the circle with radius $r_2 = \overline{QA}$.

$$r_2 = 2r_1 = \frac{2}{3}R$$

Let P be the center of the circle of radius $r_3 = \overline{PT}$.

$$\overline{OQ} = r_1, \overline{QP} = r_2 + r_3 = \frac{2}{3}R + r_3, \overline{OP} = R - r_3$$

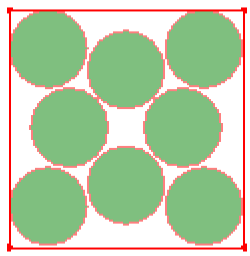
Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{QOP}$:

$$\left(\frac{2}{3}R + r_3\right)^2 = \left(\frac{1}{3}R\right)^2 + (R - r_3)^2$$

Simply put, $r_3 = \frac{1}{5}R$

The ratio between the radii is:

$$\frac{r_3}{r_1} = \frac{3}{5}$$



December 25-26: To pack eight equal circles in a square, you have to place them as in the figure, (tested by Schaer in 1964). Find the ratio of the radius of a circle to the side of the square.

Solution: Let the square ABCD of side $\overline{AB} = c$ and center O.

$$\overline{BD} = c\sqrt{2}, \overline{OD} = \frac{\sqrt{2}}{2}c$$

Let E, F, G, H, I, J, K, L centers of six circles of radius r.
The centers I, J, K, L, form a square with side 2r.

$$\overline{OT} = r$$

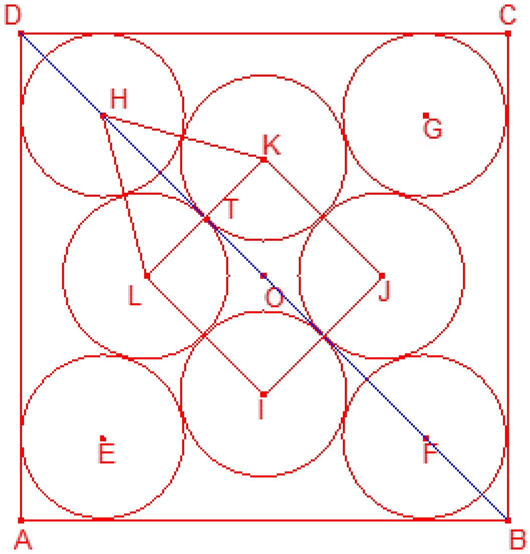
The centers K, L, H, form an equilateral triangle with side 2r.

$$\overline{HT} = r\sqrt{3}$$

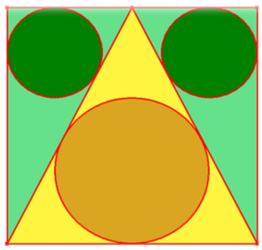
$$\overline{DH} = r\sqrt{2}$$

$$\overline{OD} = \frac{\sqrt{2}}{2}c = r + r\sqrt{3} + r\sqrt{2}$$

$$\frac{c}{r} = \sqrt{2}\left(1 + \sqrt{3} + \sqrt{2}\right) = 2 + \sqrt{2} + \sqrt{6}$$



December 29-30: Three triangles have been formed by joining the midpoint of one side of a square with the other vertices of the square. The circles inscribed in the three triangles have been drawn. Calculate the ratio between the radii of the circles



Solution: Let ABCD be the square with side $\overline{AB} = c$. Let M be the midpoint of the side $\overline{CD}$. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ADM}$

$$\overline{AM} = \frac{\sqrt{5}}{2}c$$

Let r be the radius of the circle inscribed in triangle $\overset{\Delta}{ABM}$. The area of the triangle $\overset{\Delta}{ABM}$ is equal to half the area of the square.

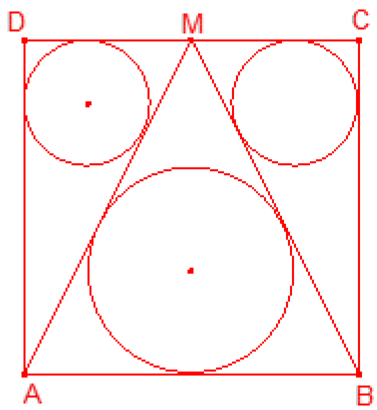
$$S_{ABM} = \frac{1}{2}c^2 = \frac{\overline{AB} + \overline{AM} + \overline{BM}}{2}r$$

$$\frac{1}{2}c^2 = \frac{c + c\sqrt{5}}{2}r$$

Solving the equation:

$$r = \frac{\sqrt{5} - 1}{4}c$$

Let s be the radius of the circle inscribed in the triangle $\overset{\Delta}{ADM}$. The area of the triangle $\overset{\Delta}{ADM}$ is equal to one fourth of the area of the square.



$$S_{ADM} = \frac{1}{4}c^2 = \frac{\overline{AD} + \overline{DM} + \overline{AM}}{2}s$$

$$\frac{1}{4}c^2 = \frac{c + \frac{1}{2}c + \frac{\sqrt{5}}{2}c}{2}s$$

$$c^2 = \frac{3 + \sqrt{5}}{4}cs$$

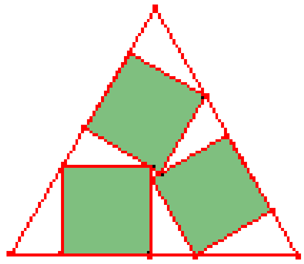
Solving the equation:

$$s = \frac{3-\sqrt{5}}{4}c$$

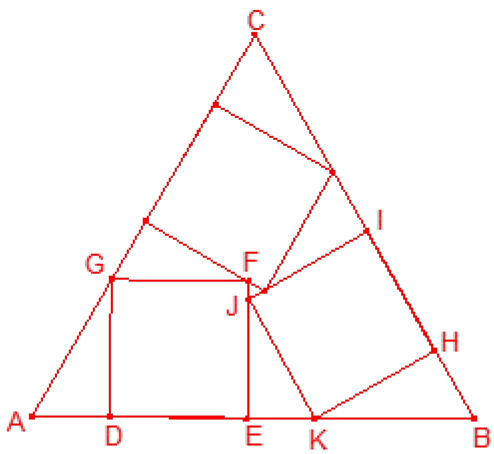
The ratio between the radii is:

$$\frac{r}{s} = \frac{\frac{\sqrt{5}-1}{4}}{\frac{3-\sqrt{5}}{4}} = \frac{1+\sqrt{5}}{2} = \Phi$$

December 31: To pack three equal squares into an equilateral triangle, you have to place them as in the figure (tested by Erich Friedman in 1997). Find the ratio of the side of the triangle to the side of the square.



Solution: Let the large equilateral triangle $\overset{\Delta}{ABC}$ with side $\overline{AB} = c$. Let the squares DEFG, HIJK of sides $\overline{DE} = x$, and $\overline{HI} = x$



$$\overline{AD} = \frac{\sqrt{3}}{3}x, \overline{EK} = \frac{1}{2}x, \overline{KB} = \frac{2\sqrt{3}}{3}$$
$$\overline{AB} = c = \left(\frac{3}{2} + \sqrt{3}\right)x$$
$$\frac{c}{x} = \frac{3}{2} + \sqrt{3}$$