SOLUTIONS JANUARY 2021

PROBLEMS FOR THE PREPARATION OF THE 3RD AND 4TH SCHOOL OLYMPIC, CALLED BY THE FESPM IN 2003. 14-16 YEARS OLD. COLLECTION MADE BY: JOSÉ COLÓN LACALLE. Retired teacher

January 1-2: I was born last century. On August 25, 2001, I was as old as the sum of the digits of the year of my birth is worth. Determine the date of my birth

Solution: Let August 25, 19xy (with x and y digits) be the subject's date of birth. We will have the statement:

$$1 + 9 + x + y = 2001 - 19xy = 2 \cdot 1000 + 1 - 1000 - 900 - 10x - y$$

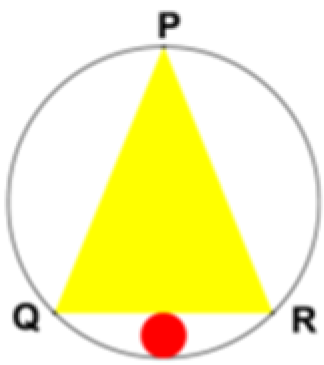
$$10 + x + y = 1000 - 899 - 10x - y; \quad 11x + 2y = 101 - 10; \quad 11x + 2y = 91$$

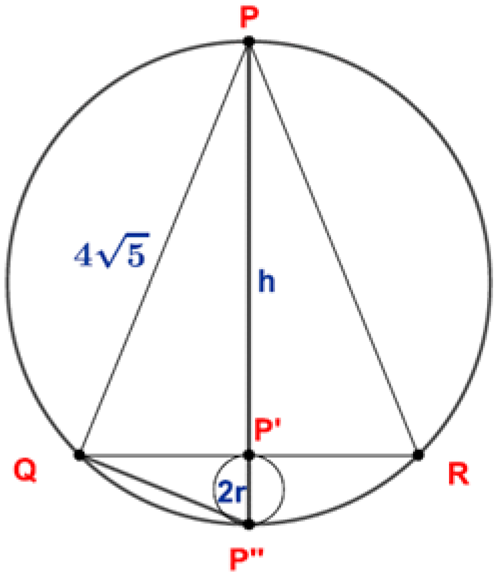
And now by trial and error:

x	$¿y = \dfrac{91 - 11x}{2} \in \mathbb{N}?$
9	No
8	No
7	Yes. y =7
6	No
5	No
4	No
3	No
2	No
1	No

Then the date of birth is **August 25, 1977**.

January 4-11: In a circle of radius 6 we inscribe an isosceles triangle ΔPQR in which PQ = PR =$4\sqrt{5}$. A second circle is tangent to the first and tangent to the base QR of the triangle, as shown in the figure. Find the radius of the small circle





Solution: Let h be the height of the isosceles triangle and r the radius of the small circle. Then:

$$h + 2r = 12\ (= \text{circumference diameter})\quad (*)$$

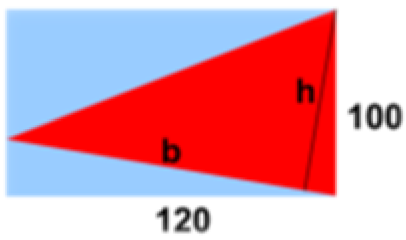
On the other hand, ΔQPP'' ≅ ΔQPP' (since they are both rectangles and have the angle at P in common). From here:

$$\frac{4\sqrt{5}}{12} = \frac{h}{4\sqrt{5}} \Rightarrow h = \frac{16 \cdot 5}{12} = \frac{20}{3}$$

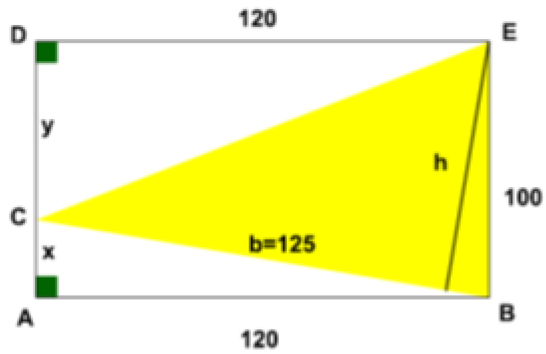
Therefore, in (*)

$$r = \frac{12 - 2r}{2} = \frac{12 - 2 \cdot \frac{20}{3}}{2} = \frac{8}{3}$$

January 5-6: In a rectangle measuring 120 long and 100 high, a triangle is inscribed as in the figure. If the base b of the triangle is 125, what is the height h?



Solution: Let x and y be the distances defined in the attached figure. We will have, x + y = 100



Applying Pythagoras to the triangle ΔABC:

$$x = \sqrt{125^2 - 120^2} = 35$$

Therefore:

$$A_{\Delta ABC} = \frac{120 \cdot 35}{2} = 2100 \quad (1)$$

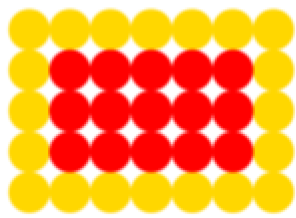
$$y = 100 - 35 = 65 \Rightarrow A_{\Delta CDE} = \frac{65 \cdot 120}{2} = 3900\ (2)$$

Of (1) and (2):

$$A_{\Delta ABC} + A_{\Delta CDE} = 2100 + 3900 = 6000$$

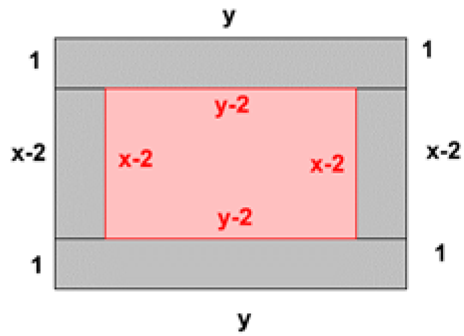
From where:

$$A_{\Delta CBE} = A_{ABDE} - (A_{\Delta ABC} + A_{\Delta CDE}) = 12000 - 6000 = 6000 = \frac{125 \cdot h}{2} \quad \Rightarrow \mathbf{h} = \frac{\mathbf{6000 \cdot 2}}{\mathbf{125}} = \mathbf{96}$$



January 7-8: A mat is formed by joining yellow circles on the outside and red circles on the inside, without overlapping. Is there a mat where the yellow circles are equal in number to the red circles?

Solution: Suppose that the mat has y circles horizontally and x vertically. So, you have 2y + 2 (x - 2) on the outside and (x - 2) · (y - 2) on the inside. It must be fulfilled, then, that:



$$2y + 2\cdot(x-2) = (y-2)\cdot(x-2)$$

$$xy - 4x - 4y + 8 = 0$$

$$y(x-4) = 4x - 8 \Rightarrow \begin{cases} x - 4 = 0 & y\cdot 0 = 4\cdot 4 - 8; 0 = 8 \text{ NO!} \\ x - 4 \neq 0 & y = \dfrac{4x-8}{x-4} = 4 + \dfrac{8}{x-4} \end{cases}$$

Then x – 4 is a divisor of 8, i.e. $x - 4 \in \{\pm1, \pm2, \pm4, \pm8\}$

If x – 4 = 1, then x = 5 and y = 4 + 8 = 12. If x – 4 = – 1, then x = 3 and y = 4 – 8 = – 4 NO.

If x – 4 = 2, then x = 6 and y = 4 + 4 = 8. If x – 4 = – 2, then x = 2 and y = 4 – 4 = 0 NO.

If x – 4 = 4, then x = 8 and y = 4 + 2 = 6. If x – 4 = – 4, then x = 0 NO

If x – 4 = 8, then x = 12 and y = 4 + 1 = 5. If x – 4 = – 8, then x = – 4 NO

Only 5x12 and 6x8 mats meet the requirement of the statement. (The other two are the previous ones turned 90°)

January 9: A brick has dimensions a, b, c. Is there a number such that if we multiply a, b, c by it, we get another brick with double area and double volume?

Solution: Let A (= 2ab + 2ac + 2bc) and V (= abc) be the area and volume of the initial brick. Let k be the number asked in the statement, which we assume exists. Then the area and volume of the new brick meet:

$$A_n = 2\cdot ka\cdot kb + 2\cdot ka\cdot kc + 2\cdot kb\cdot kc = k^2\cdot(2ab + 2ac + 2cb) = k^2\cdot A = 2A \Rightarrow k^2 = 2 \Rightarrow k = \sqrt{2}$$

$$V_n = ka\cdot kb\cdot kc = k^3\cdot V = 2\cdot V \Rightarrow k^3 = 2 \Rightarrow k = \sqrt[3]{2}$$

later on:

$$\sqrt{2} = \sqrt[3]{2}$$

Which is absurd. Therefore, some initial assumption or some step in the proof is a falsehood. Since all the steps are correct, the only assumption we have made (that k exists) is false.

January 12-13: Aitana is in charge of marking the books in the bookstore. According to her she received several books on Monday and marked some of them. On Tuesday she received as many new books as she had not marked on Monday and marked 12. On Wednesday she received 14 more books than Monday and marked twice as many books as Monday. On Thursday she received twice as many books as she had marked on Wednesday and marked 10. On Friday she received 4 books and marked 14 less books than she had received on Wednesday. On Saturday she marked the 20 remaining books of the week. Is it possible, or is she wrong?

Solution: If x (y) is the number of books received (marked) on Monday, we have:

	received	marked	remain to be marked	condition
Monday	x	y	x – y	$x - y \geq 0$
Tuesday	x – y	12	2x – 2y – 12	$x - y \geq 6$
Wednesday	x + 14	2y	3x – 4y + 2	$3x - 4y \geq -2$
Thursday	4y	10	3x – 8	$3x \geq 8$
Friday	4	x	2x – 4	$2x - 4 \geq 0$
Saturday	0	20	0	
total	3x+3y+18	x + 3y + 42		

We have two conditions that must be met: the total received must be equal to the total marked or those that remain to be marked on Friday must coincide with those marked on Saturday. We will have:

$$3x + 3y + 18 = x + 3y + 42; \quad 2x = 24; \quad \mathbf{x = 12}$$

$$2x - 4 = 20; \quad \mathbf{x = 12}$$

There is no information about y, except that provided by the last column of the table, which is reduced to:

$$\left.\begin{array}{r} \left.\begin{array}{r} 12 \geq y \\ 12 - y \geq 6 \end{array}\right\} \quad 6 \geq y \\ 3 \cdot 12 - 4y \geq -2 \quad 9{,}5 \geq y \end{array}\right\} \Rightarrow \quad 6 \geq y \geq 0$$

The statement is true if 12 books are received on Monday and any quantity between 0, 1, 2, 3, 4, 5 or 6 is marked

January 14: If a string is cut into 20 cm pieces, there is a 15 cm piece left over. If the string was three times the length, would there be any leftovers? How many cm?

Solution: Let L be the length of the string. We will have the statement:

$$20x + 15 = L \;\Rightarrow\; L - 15 = \dot{\widehat{20}} \;\Rightarrow\; 3 \cdot (20x + 15) = 60x + 45 = 3x \cdot 20 + 20 + 20 + 5 = (3x + 2) \cdot 20 + 5$$

Then, if the string is triple in length, there will be triple the number of pieces plus two, 20 cm long and there will be a 5 cm piece left over.

January 15-16: A referee chooses three hats simultaneously from a set of three white and two black. Three men seated, lined up one after the other, and all looking in the same direction (so that each one can only see the hat in front of him) close their eyes while one of the chosen hats is placed on them. Hats not chosen are hidden from view. The referee asks the third in line if he knows the color of his hat and he replies that he does not know. He asks the one sitting in the center and also replies that he does not know. So the first one says that his is white. How could he deduce it?

Solution: We will represent by triples

(color of the hat of the third, color of the hat of the second, color of the hat of the first)

the results of the experiment.

If the third man (looking at the hats of the first two) cannot guarantee the color of his hat, it is because he sees two white hats or one white and one black, because if he saw two black hats he could guarantee that his hat is necessarily white. Then the results compatible with the fact that the third party cannot ensure the color of his hat are:

(, W, W)

(, W, B)

(, B, W)

The second cannot guarantee the color of his hat either, which means that he sees (the color of the first's hat) a white hat, (if he saw a black hat then he could guarantee that he wears a colored hat White). Then if the third and second cannot guarantee the color of his hat, the first can guarantee that his hat is white.

January 18-19: If we write all the natural numbers, without any separation between them, from 1 to 2021, we obtain a number with many digits:

1234567891011121314151 6 ······ 201920202021

How many digits does this number have? What is the 2002 figure on the left?

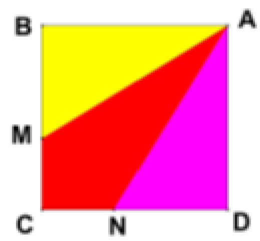
Solution: We have the following picture:

numbers	how many numbers?	figures by number	busy places	accumulated
1, ············ , 9	(9 – 0 =) 9	1	9·1 = 9	9
10, ········· , 99	(99 – 9 =) 90	2	90·2 = 180	189
100, ···· , 999	(999 – 99 =) 900	3	900·3 = 2700	2889
1000, ·· , 1999	(1999 – 999 =) 1000	4	1000·4 = 4000	6889
2000, ·· , 2021	(2021 – 1999 =) 22	4	22·4 = 88	6977

Then the number has 6977 digits.

Let's see which figure is ranked 2002 in the number from the left. As the first 189 digits (from the left) correspond to numbers with one or two digits, we will have that the place 2002 corresponds to the (2002 - 189 =) 1813 place of the numbers with three digits. Since 1813 = 604 · 3 +1, we will have that the 2002 digit of the number corresponds to the first digit of the number that occupies the 605 place (= 604 + 1) of three digits. That is, the figure 2002 is the first of the number (605 + 99 =) 704, that is, the figure 2002 is a 7.

January 20-27: Three brothers have inherited a square field that they must divide as indicated in the figure, because in A there is a well that everyone wants to use. Where must M and N be so that the surfaces of the triangles ΔABM and ΔAND and the quadrilateral AMCN are equal?

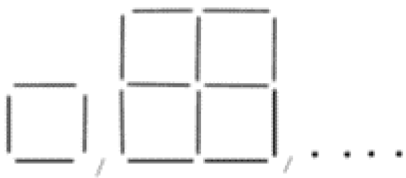


Solution: Let l be the side of the square. It is required that: $A_{\Delta ABM} = A_{AMCN} = A_{\Delta AND}$. By symmetry, it must be satisfied that $2 \cdot A_{\Delta ACN} = A_{\Delta AND}$. Therefore:

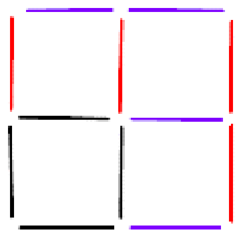
$$2 \cdot \frac{CN \cdot l}{2} = \frac{ND \cdot l}{2} \quad \Rightarrow \quad CN = \frac{ND}{2} \quad \Rightarrow \quad 2 \cdot CN = ND \quad \Rightarrow \quad l = CN + ND = 2CN + CN = 3CN$$

$$\Rightarrow \quad \mathbf{CN = CM = \frac{l}{3}}$$

January 21-22: Grids of side 1, 2, ... are being built with sticks of length one. How many toothpicks will we need to make a grid with side n?



Solution 1: The first grid consists of 4 toothpicks. The second grid consists of the 4 of the first and 4 (= 2 · 2) vertical (red) and 4 (= 2 · 2) horizontal (purple) sticks are added; that is, (2 · 2 + 2 · 2 = 2 (2 + 2) =) 2 · 4 toothpicks are added. The third grid consists of those in the second to which are added 6 (= 3 · 2) horizontals and 6 (= 3 · 2) verticals; that is, add (3 · 2 + 3 · 2 = 3 (2 + 2) =) 3 · 4 toothpicks. The fourth grid consists of those of the third grid, to which are added 8 (= 4 · 2) horizontals and 8 (= 4 · 2) verticals; that is, (4 · 2 + 4 · 2 = 4 (2 + 2) =) 4 · 4 toothpicks are added. And so on.



Namely:

$a_1 = (1 \cdot 4 + 0) = 4$

+8

$a_2 = (2 \cdot 4 + 4) = 12$ +4

+12 (*)

$a_3 = (3 \cdot 4 + 12) = 24$ +4

+16

$a_4 = (4 \cdot 4 + 24) = 40$

d_n is an arithmetic progression of difference 4 $\Rightarrow d_n = 8 + 4(n - 1) = 4 + 4n$

Therefore:

$$\mathbf{a_n} = 4 + \sum_{k=1}^{n-1} d_k = 4 + \frac{d_1 + d_{n-1}}{2}(n-1) = 4 + \frac{8 + 4 + 4(n-1)}{2}(n-1) = 4 + 6(n-1) + 2(n-1)^2$$
$$= 4 + 6n - 6 + 2n^2 - 4n + 2 = 2n^2 + 2n = \mathbf{2n(n+1)}$$

Solution 2: When we realize in (*) that a_n is a second order arithmetic progression, we will have:

$$\left.\begin{aligned} a_1 &= 4 \\ a_2 &= 12 \\ a_3 &= 24 \end{aligned}\right\} \Rightarrow a_n = An^2 + Bn + C$$

For n = 1, we will have: 4 = A + B + C (1)

For n = 2, we will have: 12 = 4A + 2B + C (2)

For n = 3, we will have: 24 = 9ª + 3B + C (3)

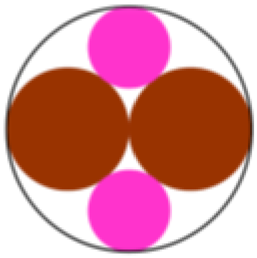
Carrying out (2) - (1), we arrive at 8 = 3A + B (4)

Carrying out (3) - (2), we arrive at 12 = 5A + B (5)

Carrying out (5) - (4), we arrive at 4 = 2A $\Rightarrow$ **A =2.**

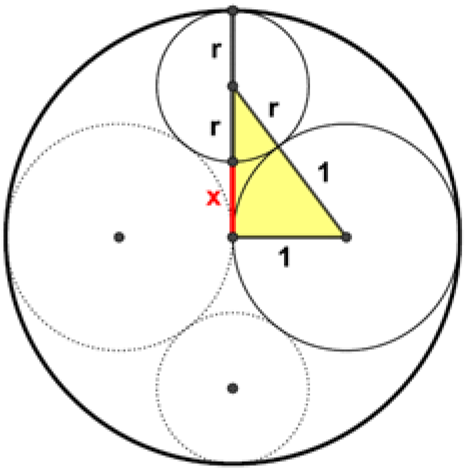
Substituting in (4), we get to **B = 2**. Substituting in (1), we get to **C = 0**. Which leads to the same solution as above.

January 23-30: In the figure there are a total of five circles, all tangent to each other. The two brown ones have radius 1. The two purple ones are the same. Find its radius.



Solution: Let r be the radius of the smallest circle. From the figure below, we will have, on the one hand:

$$x + 2r = 2 \qquad (1)$$



And applying the Pythagorean theorem to the shaded right triangle:

$$(x + r)^2 + 1^2 = (1 + r)^2 \qquad (2)$$

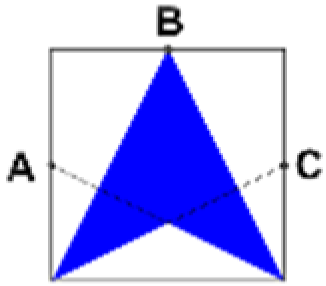
From (1) we will have, isolating x:

$$x = 2 - 2r$$

And, substituting in (2), we get to:

$$(2 - r)^2 + 1 = (1 + r)^2 \Rightarrow 4 - 4r = 2r \Rightarrow 4 = 6r \Rightarrow \mathbf{r = \frac{2}{3} m}$$

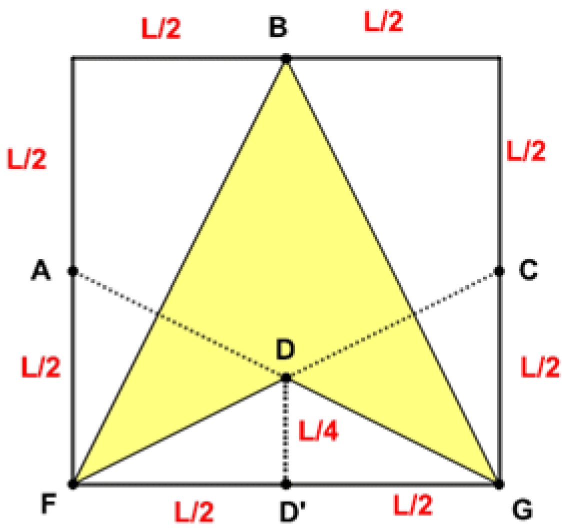
January 25: In the figure there is a square with side L. A, B, and C are midpoints. Find area and perimeter of the blue zone



Solution: In the first place we have ΔFDD' ≅ ΔFCG (well, look at the figure below, both triangles are in Thales position). Therefore:

$$\frac{DD'}{\frac{L}{2}} = \frac{\frac{L}{2}}{L} \Rightarrow DD' = \frac{\frac{L^2}{4}}{L} = \frac{L}{4}$$

S



For areas, we have:

$$\left.\begin{aligned} A_{\Delta FBG} &= \frac{1}{2} L \cdot L = \frac{L^2}{2} \\ A_{\Delta FDG} &= \frac{1}{2} L \frac{L}{4} = \frac{L^2}{8} \end{aligned}\right\} \Rightarrow \mathbf{A_{FBGD}} = A_{\Delta FBG} - A_{\Delta FDG}$$

$$= \frac{L^2}{2} - \frac{L^2}{8} = \frac{\mathbf{3 \cdot L^2}}{\mathbf{8}}$$

For the perimeter, we have:

$$\mathbf{P} = 2 \cdot FB + 2 \cdot FD = 2\sqrt{\frac{L^2}{4} + L^2} + 2\sqrt{\frac{L^2}{4} + \frac{L^2}{16}}$$

$$= 2\frac{\sqrt{5}L}{2} + 2\frac{\sqrt{5}L}{4} = \mathbf{3\frac{\sqrt{5}L}{2}}$$

<u>January 26:</u> A stadium has a capacity of 25,000 spectators. One day, with the stadium almost full, $\mathbf{15,\hat{5}\%}$ of the spectators were in the South stands and $\mathbf{24,\widehat{524}\%}$ were women. How many spectators could there be in the stadium?

<u>Solution:</u> Let x be the number of attendees. We know that x < 25000. We will also have that 15.5 de% of x is a natural number, so:

$$15,\hat{5}\% \text{ de } x = \frac{15,\hat{5}}{100} \cdot x = \frac{\frac{155 - 15}{9}}{100} \cdot x = \frac{140}{900} \cdot x = \frac{7}{45} \cdot x \in \mathbb{N}$$

And since 7 does not divide 45, x must be a multiple of 45.

Similarly, $24,\widehat{524}\%$ of x is a natural number, so:

$$24,\widehat{524}\% \text{ de } x = \frac{24,\widehat{524}}{100} \cdot x = \frac{\frac{24524 - 24}{999}}{100} \cdot x = \frac{245}{999} \cdot x \in \mathbb{N}$$

And since 245 and 999 have no common factors, x must be a multiple of 999.

Since x must be a multiple of 45 and a multiple of 999, it must be a multiple of the LCM (45, 999) (= LCM ($3^2 \cdot 5$; $3^3 \cdot 37$) = $3^3 \cdot 37 \cdot 5$) = 4995. The smallest possible value of x is 4995. The other possible values of x are multiples of 4995, that is: (4995 · 2 =) 9990; (4995 · 3 =) 14985; (4995 · 4 =) 19980; (4995 · 5 =) 24975. Since the subsequent multiples exceed 25000, this is the last possible solution. As, in addition, the statement says that the stadium was almost full, we concluded that there were **24,975 attendees**.

<u>January 28:</u> Dani had to add up all the four-digit palindromic numbers, but forgot to add one. If he got 490776, which one did he forget?

<u>Solution:</u> We will designate by $\overline{xyyx}$ the palindromic number that has the digit x (≠ 0) in the units of a thousand and in the units and the digit y in the hundreds and tens. Let's calculate:

$$\sum_{x,y} \overline{xyyx} = \sum_{x=1}^{9} \left(\sum_{y=0}^{9} \overline{xyyx} \right)$$

We have:

$$\sum_{y=0}^{9} \overline{xyyx} = \sum_{y=0}^{9} (x \cdot 1000 + y \cdot 100 + y \cdot 10 + x)$$

$$= \sum_{y=0}^{9} (x \cdot 1001 + y \cdot 110) = x \cdot 1001 \cdot 10 + 110 \cdot \sum_{y=0}^{9} y = 10010 \cdot x + 110 \cdot \frac{0+9}{2} \cdot 10$$

$$= 10010x + 45 \cdot 110$$

$$\sum_{x,y} \overline{xyyx} = \sum_{x=1}^{9} \left(\sum_{y=0}^{9} \overline{xyyx} \right)$$

$$= \sum_{x=1}^{9} (10010x + 45 \cdot 110) = 10010 \cdot \sum_{x=1}^{9} x + 45 \cdot 110 \cdot 9 = 10010 \cdot \frac{1+9}{2} \cdot 9 + 45 \cdot 990$$

$$= 10010 \cdot 45 + 45 \cdot 990 = 45(10010 + 990) = 45 \cdot 11000 = 49500$$

Therefore, the number Dani forgot to add is (49500 – 490776 =) 4224

<u>January 29:</u> How many triples of naturals different from the unit (a, b, c), are there such that: **$a \cdot b \cdot c = 7^{39}$**?

<u>Solution:</u> Since 7 is prime, obviously $a = 7^{\alpha}$, $b = 7^{\beta}$ y $c = 7^{\delta}$. Since a, b, and c must be different from unity, the exponents must be positive. Therefore, there are as many triples (a, b, c) as triples of positive exponents $\alpha + \beta + \delta = 39$. Furthermore:

$$(7, 7, 7^{37}) \neq (7, 7^{37}, 7)$$

That is, the order matters. We proceed to count the triples

α	β	δ	
1	1	37	
1	2	36	
1	3	35	
.	.	.	
.	.	.	37
.	.	.	
1	36	2	
1	37	1	
2	1	36	
2	2	35	
2	3	34	
.	.	.	
.	.	.	36
.	.	.	
2	35	2	
2	36	1	

3	1	35	
3	2	34	
.	.	.	35
.	.	.	
3	34	2	
3	35	1	
.	.	.	
.	.	.	
.	.	.	
36	1	2	2
36	2	1	
37	1	1	1

In summary:

$$1 + 2 + 3 + \cdots + 36 + 37 = \frac{1 + 37}{2} \cdot 37 = 703$$