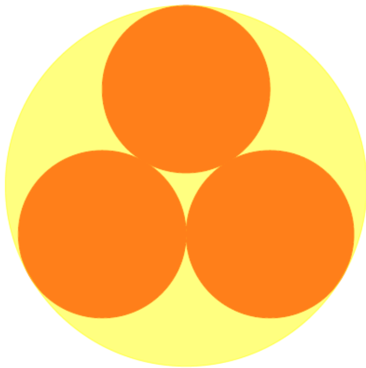
SOLUTIONS FEBRUARY 2021

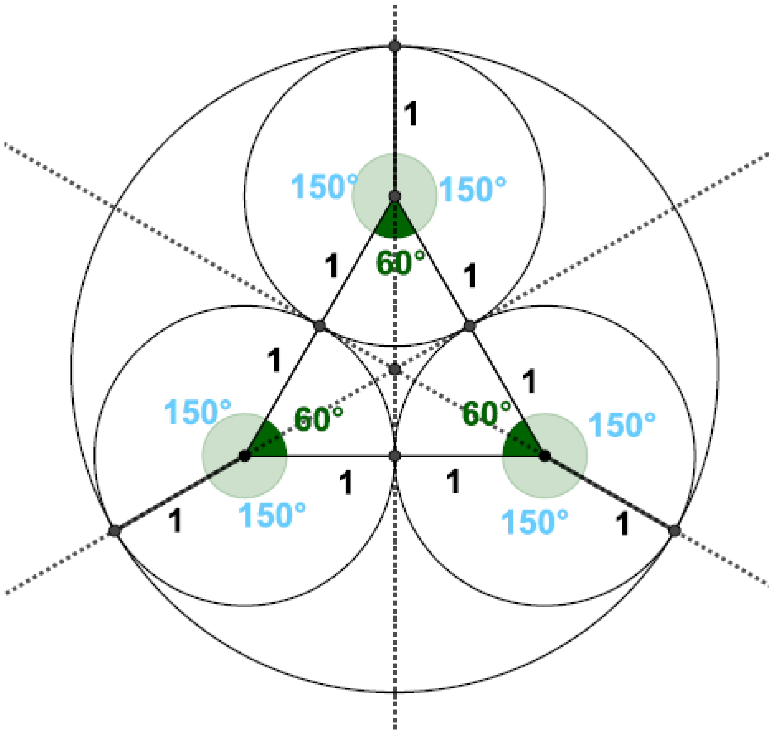
PROBLEMS OF THE CMO 1972 AND 1973. (CANADIAN MATHEMATICAL OLYMPIAD). 16-18 YEARS. OME PREPARATION. ORGANIZATION: RAFAEL MARTÍNEZ CALAFAT. Retired teacher
https://cms.math.ca/Competitions/CMO/

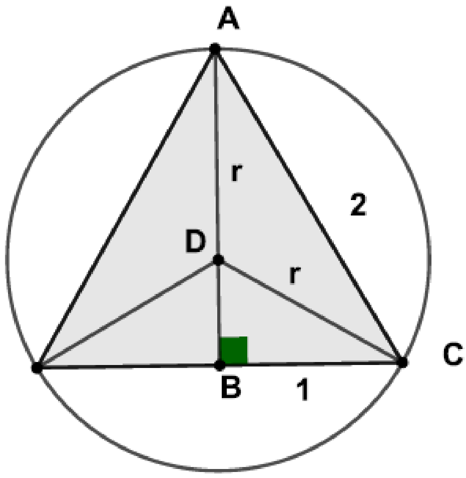
February 1-2: Let be given three circles of unit radius, each one tangent to the exterior of the other two. Find the radius of the circle circumscribing the three initial circles



Solution: The centers of the three circles of unit radius form a triangle with sides 2, and therefore equilateral. Hence, the interior angles to the triangle are 60° and (by symmetry of the large circle) the exterior angles are 150°.

The radius of the large circle (R) is the radius of the circle formed by the three vertices of the triangle (r) plus 1





So let's find the radius of this inner circle. In the figure on the right we have that ΔABC is a triangle 30°-60°-90°, so its sides are in the ratio $1:\sqrt{3}:2$. Also (well, for basic geometry, or for the similarity ΔABC ≅ ΔDCB):

$$r = \frac{2}{3}AB = \frac{2}{3}\sqrt{3} \quad \Rightarrow \quad R = 1 + \frac{2}{\sqrt{3}} = \frac{\sqrt{3}+2}{\sqrt{3}} = \frac{3+2\sqrt{3}}{3}$$

February 3: Prove that 10201 is composed of any base greater than 2. Prove that 10101 is composed of any base.

Solution: For the first part, the base of the numbering system is required to be greater than 2 for the digit 2 in the expression of the given number to make sense. We will have:

$$10201_a = 1 \cdot a^4 + 2 \cdot a^2 + 1 = (a^2+1)^2 = (101_a)^2 = 101_a \cdot 101_a$$

Then 10201_a is compound.

For the second part of the problem, we must show that 10101a can be factored. How:

$$10101_a = a^4 + a^2 + 1$$

we can consider the previous expression as a polynomial in a of the fourth degree (with the main coefficient of unity), which is factored as a product of polynomials of degree 1 or of degree 2 with negative discriminant. Since there are no real roots, there are no polynomials of degree 1

$$\left(x^4 + x^2 + 1 = 0 \quad \Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4}}{2} \notin \mathbb{R} \right)$$

Let's put:

$$a^4 + a^2 + 1 = (a^2 + Aa + B) \cdot (a^2 + Ca + 1) = a^4 + (C + A)a^3 + (D + AC + B)a^2 + (AD + BC)a + BD$$

With what we have:

$$\left.\begin{array}{c} C + A = 0 \\ D + AC + B = 1 \\ AD + BC = 0 \\ BD = 1 \end{array}\right\}$$

From the first C = - A, and substituting in the other three equations, we arrive at:

$$\left.\begin{array}{c} D - A^2 + B = 1 \\ AD - AB = 0 \\ BD = 1 \end{array}\right\} \Rightarrow \; A(D - B) = 0$$

If A = 0, then C = 0 and the system reduces to:

$$\left.\begin{array}{c} D + B = 1 \\ DB = 1 \end{array}\right\} \text{that has no solution in the real ones.}$$

If $A \neq 0$, then D = B and in this case:

$$\left.\begin{array}{c} 2D - A^2 = 1 \\ B^2 = 1 \end{array}\right\}$$

From the second equation, we have $B = \pm 1$.

If B = 1, then D = 1, which leads (in the first equation) to $A = \pm 1$ and $C = \mp 1$

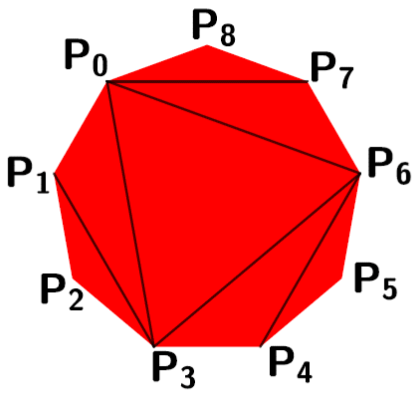
If $B = -1 = D$, then $-2 - 1 = A^2$, which is impossible.

Then:

$$\begin{aligned} a^4 + a^2 + 1 = (a^2 - a + 1) \cdot (a^2 + a + 1) \;\Rightarrow\; 10101_a &= \big((a-1)a + 1\big) \cdot (a^2 + a + 1) \\ &= (a-1)1_a \cdot 111_a \end{aligned}$$

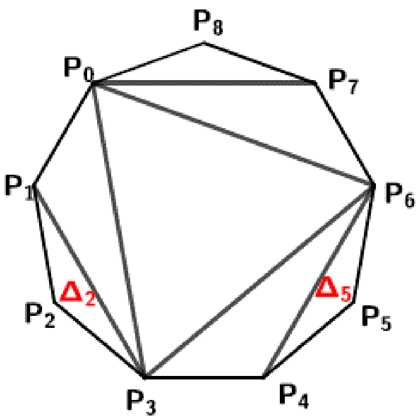
(Note: The number $(a - 1)1_a$ is the number that, expressed based a, has the digit a - 1 in the powers of a^1 and the digit 1 in the power of a^0)

February 4-5: The figure shows a convex polygon with 9 vertices. The 6 diagonals drawn dissect it into 7 triangles: P_0, P_1, P_2, ⋯, P_6, P_7, P_8. In how many ways can these triangles be named with the symbols Δ_1, Δ_2, Δ_3, Δ_4, Δ_5, Δ_6, Δ_7 so that the triangle Δ_i has P_i $\forall i \in \{1, 2, 3, 4, 5, 6, 7\}$ as its vertex. Justify your answer.



Solution: We have to name the triangles with the labels Δ_i so that a triangle is labeled Δ_i if and only if P_i is a vertex of the triangle. As there is only one triangle with vertex P_2 (P_5) only those triangles can receive the labels Δ_2 (Δ_5) (figure 1). Before this assignment, there were two triangles with vertex P_1 (P_4) of which only one remains.

Figure 1



This triangle is labeled Δ_1 (Δ_4) (figure 2). Of the triangles with vertex P_3 only one remains, which is labeled Δ_3 (figure 3). There remains a triangle with vertex P_6 that is labeled Δ_6. Finally, there is only one triangle that has P_7 as its vertex, which is labeled Δ_7

Figure 2

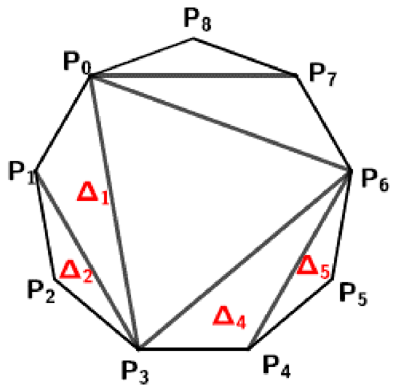
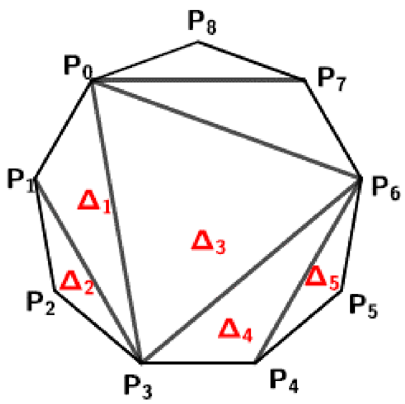
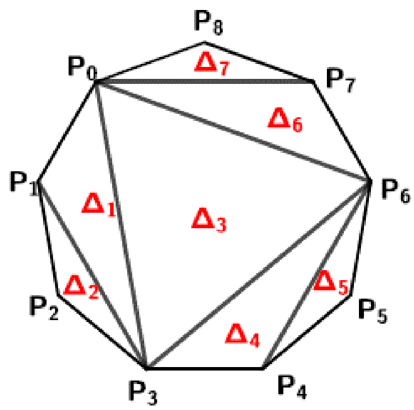


Figure 3



There is only one way to label the triangles so that the triangle Δ_i has as its vertex P_i $\forall i \in \{1, 2, 3, 4, 5, 6, 7\}$

Figure 4



February 6: Find the largest integer that satisfies the two inequalities:

$$4x + 13 < 0$$

$$x^2 + 3x > 16$$

Solution: For the linear inequality we have:

$$4x + 18 \leq 0 \quad \Leftrightarrow \quad x \leq -\frac{13}{4} \quad \Leftrightarrow \quad x \in \left]-\infty; -\frac{13}{4}\right[$$

For the quadratic inequality, we have:

$$x^2 + 3x > 16 \quad \Leftrightarrow \quad x^2 + 3x - 16 > 0$$

Let $y = x^2 + 3x + 16$. Its graph is a parabola directed upwards ($a = 1 > 0$) and as:

$$x^2 + 3x - 16 = 0 \quad \Leftrightarrow \quad x = \frac{-3 \pm \sqrt{73}}{2}$$

we will have:

$$x^2 + 3x > 16 \quad \Leftrightarrow \quad x \in \left]-\infty; -\frac{3 + \sqrt{73}}{2}\right[\cup \left]\frac{-3 + \sqrt{73}}{2}; +\infty\right[$$

The two inequalities are satisfied in $x \in \left]-\infty; -\frac{3+\sqrt{73}}{2}\right[$. Therefore, the largest integer that satisfies the two inequalities is – 6

February 8: Prove that the equation:

$$x^3 + 11^3 = y^3$$

has no solutions to positive integers.

Solution 1: https://ca.wikipedia.org/wiki/Darrer_teorema_de_Fermat

Fermat's Last Theorem, also known today as the Wiles-Fermat theorem, states that the Diophantine equation

$$x^n + y^n = z^n$$

it has no integer solution for n> 2 and where x, y and z are nonzero.

It is one of the most famous theorems in the history of mathematics and until 1995 a proof was not available (and, therefore, it was strictly called Fermat's conjecture). Note that when n = 2 the equation is equivalent to the Pythagorean theorem and obviously has infinite solutions.

The French mathematician Pierre de Fermat was the first to propose the theorem, but unfortunately the proof that he supposedly made has never been found. Fermat only wrote in a margin of his copy of Diophantus's Arithmetic the statement of the theorem and the statement that he had found a proof of the theorem. In his own words:

Cubum autem in duos cubos, aut cuadrados- quadratum in duos cuadrados- quadratos, te generalidad nulamente in infinitum ultra quadratum potestades in duos eiusdem nominis haces este Divide cuius rey Demonstration mirabilis sane detexi. Hanc marginado exiguitas non caperet

namely,

«It is impossible that a cube is the sum of two cubes, that a fourth power is the sum of two fourth powers and, in general, that any number that is a power greater than two is the sum of two powers of the same value. I have found a truly wonderful proof of this proposition, but this margin is too narrow to fit. "

Fermat's claim immediately became a problem that many mathematicians tried to solve. Little by little partial proofs emerged (for example, Sophie Germain proved the theorem in the case where n is a prime number and 2n + 1 is also a prime number) or proofs of theorems associated with it. The theorem was also proved for very determined values of n: Euler proved it for n = 3, Fermat himself recorded his proof for n = 4, Legendre and Dirichlet for n = 5 and the latter also for n = 14.

In 1993 Andrew Wiles announced the general proof of the theorem, a proof that turned out to be wrong, but which he himself corrected towards the end of 1994. [1] With this proof, which involves the use of elliptic functions and Galois representations, one of the most famous Mathematical problems remained closed. However, it is worth asking whether Fermat actually got a proof of the theorem from him and, if so, what method he used, since the path followed by Wiles uses mathematical tools that did not exist in Fermat's time.

Solution 2: We will have:

$$x^3 + 11^3 = y^3; \quad y^3 - x^3 = 11^3; \quad (y - x)\cdot(y^2 + yx + x^2) = 11^3$$

And due to the uniqueness of the factorial decomposition in prime numbers, the following cases fit:

$$\left.\begin{matrix} y - x = 1 \\ y^2 + xy + x^2 = 11^3 \end{matrix}\right\}(1) \quad \left.\begin{matrix} y - x = 11 \\ y^2 + xy + x^2 = 11^2 \end{matrix}\right\}(2) \quad \left.\begin{matrix} y - x = 121 \\ y^2 + xy + x^2 = 11 \end{matrix}\right\}(3) \quad \left.\begin{matrix} y - x = 11^3 \\ y^2 + xy + x^2 = 1 \end{matrix}\right\}(4)$$

For the case (1), we have solving y from the first equation and substituting in the second:

$$(1 + x)^2 + x(x + 1) + x^2 = 1331; \quad 3x^2 + 3x - 1330 = 0; \quad x = \frac{-3 \pm \sqrt{15969}}{6} \notin \mathbb{Z}$$

For the case (2), we have solving y from the first equation and substituting in the second

$$(11 + x)^2 + x(x + 11) + x^2 = 121; \quad 3x^2 + 33x = 3x\cdot(x + 11) = 0; \quad \begin{cases} 3x = 0; \quad x = 0 \notin \mathbb{Z}^+ \\ x = -11 \notin \mathbb{Z}^+ \end{cases}$$

For the case (3), we have solving y from the first equation and substituting in the second:

$$(121 + x)^2 + x(x + 121) + x^2 = 11; \quad 3x^2 + 363x + 14630 = 0; \quad x = \frac{-363 \pm \sqrt{-43791}}{6} \notin \mathbb{R}$$

For the case (4), we have solving y from the first equation and substituting in the second:

$$(1331 + x)^2 + x(x + 1331) + x^2 = 1; \quad 3x^2 + 2992x + 1771560 = 0; \quad x = \frac{-2992 \pm \sqrt{-12306656}}{6} \notin \mathbb{R}$$

With this, we have that the only solutions in $\mathbb{Z}$ of the given equation is x = – 11 e y = 0

February 9: If a and b are different reals. Prove that there are integers m and n such that

am + bn < 0

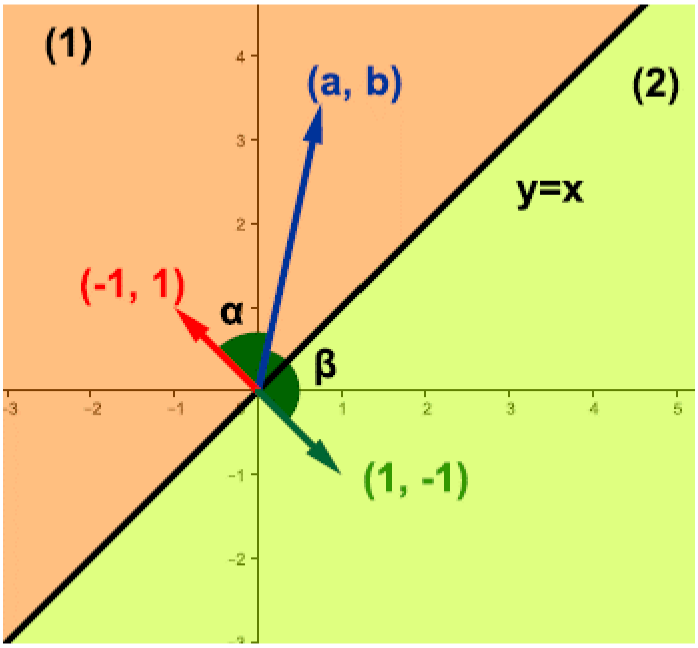
bm + an > 0

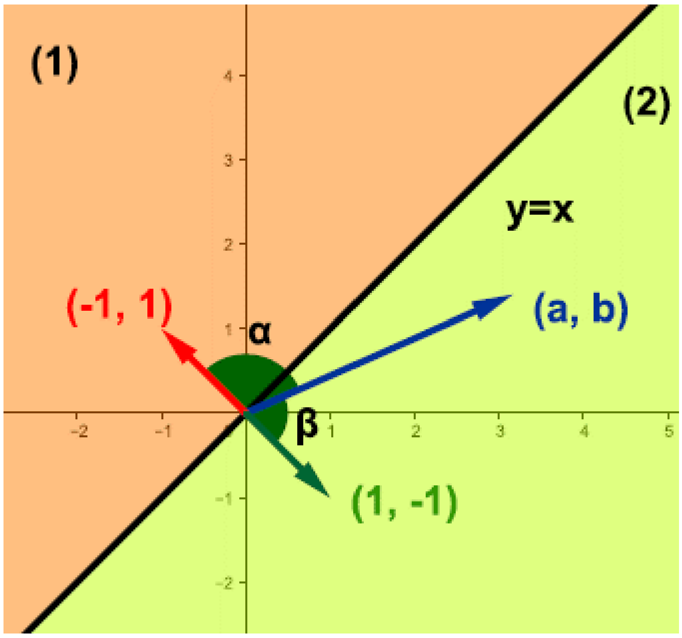
Genesis of the solution: Recall that given two vectors $\vec{u} = (a, b)$ and $\vec{v} = (n, m)$, we define:

$$\vec{u}\cdot\vec{v} = \begin{cases} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\left(\widehat{\vec{u},\vec{v}}\right) \\ = an + bm \end{cases}$$

Since $\|\vec{u}\|$, $\|\vec{v}\| \geq 0$, the sign of $\vec{u}\cdot\vec{v}$, depends on the sign of $\cos\left(\widehat{\vec{u},\vec{v}}\right)$, which, in turn, depends on $\widehat{\vec{u},\vec{v}}$. If $\widehat{\vec{u},\vec{v}} \in]0°, 90°[\quad \Rightarrow \quad \vec{u}\cdot\vec{v} > 0$. If $\widehat{\vec{u},\vec{v}} \in]90°, 180°[\quad \Rightarrow \quad \vec{u}\cdot\vec{v} < 0$. With this, given $\vec{u} = (a, b)$, (a ≠ b) we have to find a vector of integer components (m, n), so that it forms less than 90° with (a, b) and such that (n, m) form more than 90° with (a, b).

Since a ≠ b, (a, b) is in region (1) or region (2) (see attached figure). If (a, b) is in the region (1) then we can take (m, n) = (1, - 1), since then (a, b) and (m, n) form an angle greater than 90° and then (a, b) · (1, - 1) = a - b < 0 and (a, b) · (- 1, 1) = b – a > 0. The same vectors are used for the case (a, b) is in the region (2), that is, if a> b, since in this case (a, b) forms an angle greater than 90° with (- 1, 1) and an angle less than 90° with (1 , - 1).





<u>Solution:</u> Given a and b. Suppose a > b, then:

0 > am +bn = (a, b)·(m, n) = (a, b)·(– 1, 1) = b – a ⇔ – b + a = (a , b)·(1, – 1) = (a, b) · (n, m) = an + bm > 0

If a < b, then:

0 < an +bm = (a, b)·(n, m) = (a, b)·(– 1, 1) = b – a ⇔ – b + a = (a , b)·(1, – 1) = (a, b) · (m, n) = am + bn < 0

<u>February 10:</u> What is the maximum number of terms of a geometric progression of naturals of ratio r > 1 that are between 100 and 1000 including both?

<u>Solution:</u> Let r be the ratio of the geometric progression. If we consider $r \in \mathbb{N}$, this is $r \in \{2, 3, 4, 5, \cdots\}$, obviously the progression must contain the number 100 (if the number 100 is exceeded, there are fewer terms up to 1000 than starting from 100). The largest number of terms comes out for $r = 2$

$$a_i = 100; \quad a_{i+1} = 200; \quad a_{i+2} = 400; \quad a_{i+3} = 800; \quad \text{(four terms)}$$

Let us now consider $r = \frac{m}{n} > 1$ (with n, m $\in \mathbb{N}$). The first term of the geometric progression that is in the considered interval must be a power as close as possible and greater than 100 of the denominator of r. How:

$$2^4 = 16; \quad 2^5 = 32; \quad 2^6 = 64; \quad 2^7 = 128$$

We choose 128 as the first element of the geometric progression that is in the considered interval. The smallest possible value for n is 2 and the value for m must be greater than 2: 3, 4, 5, ⋯. Among these, since we want there to be as many terms as possible within [100; 1000], we choose m = 3 as optimal. We will therefore have, $r = \frac{3}{2}$

$$a_i = 128; \quad a_{i+1} = 192; \quad a_{i+2} = 288; \quad a_{i+3} = 432; \quad a_{i+4} = 648; \quad a_{i+5} = 972; \text{(six terms)}$$

If we try with the following denominator: 3 (and since r> 1 and as low as possible to contain the largest number of terms within the range 100-1000) and with the numerator 4, that is, with $r = \frac{4}{3}$ we will have: Again the first term of the geometric progression must be greater than or equal to 100 and a power of the denominator 3

$$3^4 = 81 \quad \Rightarrow \quad \begin{cases} 3^5 = 243 \\ 2 \cdot 3^4 = 162 \end{cases}$$

$$r = \frac{4}{3}; \quad a_i = 243; \quad a_{i+1} = 324; \quad a_{i+2} = 432; \quad a_{i+3} = 576; \quad a_{i+4} = 768; \quad \text{(five terms)}$$

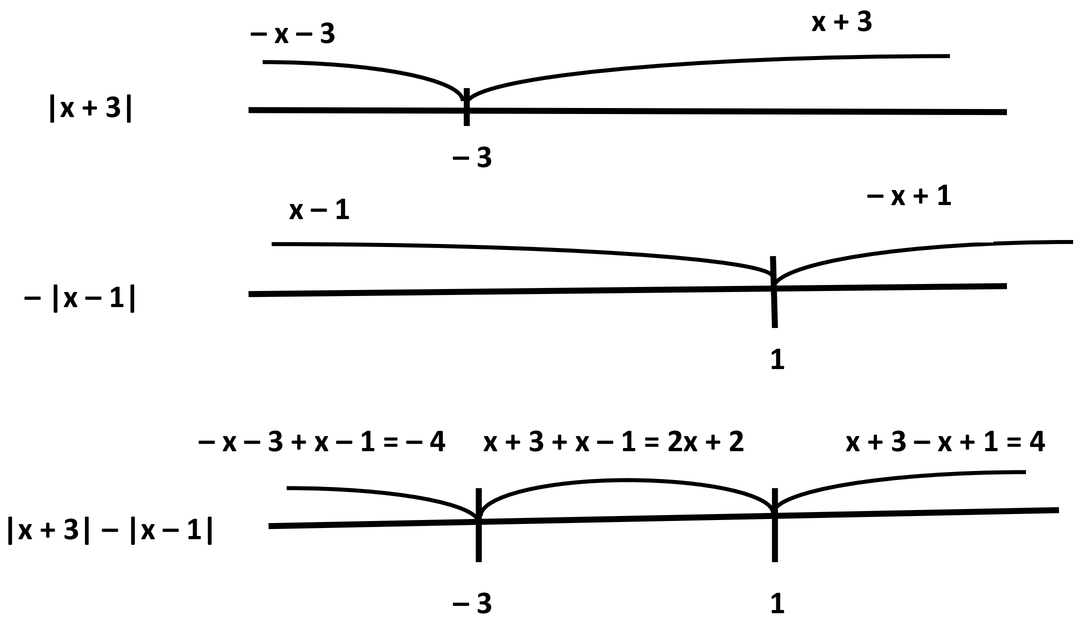
$$r = \frac{4}{3}; \quad a_i = 162; \quad a_{i+1} = 216; \quad a_{i+2} = 288; \quad a_{i+3} = 384; \quad a_{i+4} = 512; \quad a_{i+5} = \frac{2048}{3} = 682,\widehat{6} \quad \text{(five terms)}$$

From among the possible r with denominator 4 we would choose the one with numerator 5. As we seek that there are as many natural terms as possible and within the range 100-1000, the first term must contain the factor 4 at least five or six times. Since 4^5 = 1024 no more cases can be considered. In short, **the geometric progression that has more natural terms within the range 100-1000 comes out with r = 3/2 and a_1 = 128 that contains six natural terms in the range 100-1000**

<u>February 11:</u> Find the reals that satisfy the equation:

$$|x + 3| - |x - 1| = x + 1$$

<u>Solution:</u> We have:



Then:

$$\text{If } x \le -3 \;\Rightarrow\; |x+3| - |x-1| = x+1 \;\Rightarrow\; -4 = x+1 \;\Rightarrow\; \mathbf{x = -5}$$

$$\text{If } x \in \,]-3; 1[\;\Rightarrow\; |x+3| - |x-1| = x+1 \;\Rightarrow\; 2x+2 = x+1 \;\Rightarrow\; \mathbf{x = -1}$$

$$\text{If } x \ge 1 \;\Rightarrow\; |x+3| - |x-1| = x+1 \;\Rightarrow\; 4 = x+1 \;\Rightarrow\; \mathbf{x = 3}$$

<u>February 12-13:</u> For any natural be:

$$\mathbf{h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$$

Prove that for n = 2, 3, 4, ···· it is true:

$$\mathbf{n} + \sum_{\mathbf{i=1}}^{\mathbf{n-1}} \mathbf{h(i)} = \mathbf{n} \cdot \mathbf{h(n)} \qquad (*)$$

Solution: Let's express the first member of (*) as rows:

As the sum is associative we can alter the order of the addends without altering the value of the sum.

We add by columns:

$$
\begin{array}{lcccccccc}
n = & 1+ & 1+ & 1+ & \dots & \dots & +1 & +1 & +1\\
h(1) = & 1 & & & & & & & \\
h(2) = & 1+ & \frac{1}{2} & & & & & & \\
h(3) = & 1+ & \frac{1}{2}+ & \frac{1}{3} & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & \\
h(n-2) & 1+ & \frac{1}{2}+ & \frac{1}{3}+ & \dots & \dots & +\frac{1}{n-2} & & \\
h(n-1) = & 1+ & \frac{1}{2}+ & \frac{1}{3}+ & \dots & \dots & +\frac{1}{n-2} & +\frac{1}{n-1} & \\
n+\sum_{i=1}^{n-1} h(i) = & n\cdot 1+ & n\cdot\frac{1}{2}+ & n\cdot\frac{1}{3}+ & \dots & \dots & +n\cdot\frac{1}{n-2} & +n\cdot\frac{1}{n-1} & +n\cdot\frac{1}{n}
\end{array}
$$

That is:

$$
\begin{aligned}
\mathbf{n} + \sum_{\mathbf{i=1}}^{\mathbf{n-1}} \mathbf{h(i)} &= n\cdot 1 + n\cdot\frac{1}{2} + n\cdot\frac{1}{3} + \cdots + n\cdot\frac{1}{n-2} + n\cdot\frac{1}{n-1} + n\cdot\frac{1}{n}\\
&= n\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right) = \mathbf{n}\cdot\mathbf{h(n)}
\end{aligned}
$$

February 15: Express 100000 as a product of two integers none of which is a multiple of 10.

Solution: We will have:

$$100000 = 10^5 = (2\cdot 5)^5 = 2^5\cdot 5^5 = 32\cdot 3125 = (-32)\cdot(-3125)$$

Any other grouping of the prime factors of 10^5 leads to expressing 10^5 as a product of integers, one of which is a multiple of 10.

February 16-17: During a certain political campaign, p different promises are made between the participating political parties. If:

1.- Several parties can make the same promise.

2.- Any two parties have at least one promise in common.

3.- There are no two parties with exactly the same promises.

Prove that there are no more than 2^{p-1} participating parties

Solution: If there are two promises: 1 and 2, then the party system that meets the three conditions of the statement are: $\{p_1, p_{12}\}$ (the subscripts refer to the promises advertised by each party). If we add another match: p_2, then the second condition is not met. That is, for two promises, there are at most $2^{2-1} = 2^1 = 2$ participating parties that meet the three conditions of the statement.

Suppose 3 promises: 1, 2 and 3, then to the maximum system of two promises: {p1, p12} we add the added promise in each subscript and we get $\{p_1, p_{12}, p_{13}, p_{123}\}$. Such a party system meets the three requirements and is the maximum because if, for example, we add another party, for example, p_{23} the second condition is not met. That is, for three promises, there are at most $2^{3-1} = 2^2 = 4$ participating parties that meet the three conditions of the statement.

Suppose four promises. Then to the maximal system for three promises $\{p_1, p_{12}, p_{13}, p_{123}\}$ we add the added promise in each subscript and we get $\{p_1, p_{12}, p_{13}, p_{123}, p_{14}, p_{124}, p_{134}, p_{1234}\}$. Such a party system meets the three requirements and is the maximum because if, for example, we add another party, for example, p23 the second condition is not met. That is, for four promises, there are at most $2^{4-1} = 2^3 = 8$ (we multiply the previous maximal system by two) participating parties that meet the three conditions of the statement.

Once we have seen the mechanism of generation of the party system, we proceed to the demonstration of the problem. Inductively.

For 2, 3 and 4 promises, we have already proven it. Suppose the hypothesis shown for {1, 2, ··, i} promises. That is, suppose that for i promises we have that the previous system generates a set of 2^{i-1} matches that meets the three requirements and that this system is maximal. Suppose we add the i + 1 promise and generate the system of double number of parties for i promises by adding the matches to the original parties adding to each of them the i + 1 promise as a subscript. This new party system meets the three requirements of the statement:

1. It is obvious.
2. Given two parties, we eliminate from them the promise i + 1 (if any of them had it) and then we have two parties from the previous system that must fulfill the second condition.
3. It is obvious.

In addition, the new system is maximal because when adding a new party, one of the three conditions of the statement is no longer met.

February 18-19: Prove that ∀n natural

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n \cdot (n+1)}$$

Prove that ∀n natural greater than 1, ∃i, j naturals such that:

$$\frac{1}{n} = \frac{1}{i \cdot (i+1)} + \frac{1}{(i+1) \cdot (i+2)} + \cdots + \frac{1}{j \cdot (j+1)}$$

Solution: The first part is obvious:

$$\frac{1}{n+1} + \frac{1}{n \cdot (n+1)} = \frac{n}{(n+1) \cdot n} + \frac{1}{n \cdot (n+1)} = \frac{n+1}{n \cdot (n+1)} = \frac{1}{n}$$

The second part is not overly complicated. For n = 2

$$\frac{1}{2}=\frac{1}{1\cdot 2}=\frac{1}{(1+1)\cdot 1} \qquad (\mathbf{i}=\mathbf{j}=\mathbf{1})$$

And for n > 2:

$$\frac{1}{n}=\frac{1}{n\cdot(n+1)}+\frac{1}{n+1}=\left\{\frac{1}{n+1}=\frac{1}{(n+1)\cdot(n+2)}+\frac{1}{n+2}\right\}=\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}+\frac{1}{n+2}$$
$$=\left\{\frac{1}{n+2}=\frac{1}{(n+2)\cdot(n+3)}+\frac{1}{n+3}\right\}$$
$$=\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}+\frac{1}{(n+2)\cdot(n+3)}+\frac{1}{n+3}=\cdots\cdots$$
$$=\left\{\begin{matrix}\text{we apply the first}\\ \text{part } n^2-2n \text{ times}\\ (n>2\ \Rightarrow\ n^2-2n>0)\end{matrix}\right\}$$
$$=\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}+\frac{1}{(n+2)\cdot(n+3)}+\cdots+\frac{1}{(n^2-n-1)\cdot(n^2-n)}$$
$$+\frac{1}{n^2-n}=\left\{\frac{1}{n^2-n}=\frac{1}{n\cdot(n-1)}\right\}$$
$$=\frac{1}{n\cdot(n-1)}+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}+\cdots+\frac{1}{(n^2-n-1)\cdot(n^2-n)}$$
$$=\left\{\begin{matrix}\mathbf{i}=\mathbf{n}-\mathbf{1}\\ \mathbf{j}=\mathbf{n^2}-\mathbf{n}-\mathbf{1}\end{matrix}\right\}=\frac{1}{i\cdot(i+1)}+\frac{1}{(i+1)\cdot(i+2)}+\cdots+\frac{1}{j\cdot(j+1)}$$

For example:

$$\frac{1}{3}=\left\{\begin{matrix}i=n-1=3-1=2\\ j=n^2-n-1=9-4=5\end{matrix}\right\}=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}=\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$$

<u>February 20:</u> Evaluate the expression:

$$\frac{\mathbf{1}}{\mathbf{\log_2 36}}+\frac{\mathbf{1}}{\mathbf{\log_3 36}}$$

<u>Solution:</u> We will have:

$$\frac{1}{\log_2 36}+\frac{1}{\log_3 36}=\frac{\log_2 2}{\log_2 36}+\frac{\log_3 3}{\log_3 36}=\left\{\begin{matrix}\text{cambio de base}\\ \log_a x=\dfrac{\log_b x}{\log_b a}\end{matrix}\right\}=\log_{36} 2+\log_{36} 3=\log_{36}(2\cdot 3)$$

$$\log_{36} 6=z\ \Leftrightarrow\ 36^z=6\ \Leftrightarrow\ z=\frac{1}{2}$$

Therefore:

$$\frac{1}{\log_2 36}+\frac{1}{\log_3 36}=\log_{36}(2\cdot 3)=\frac{1}{2}$$

<u>Wrong demonstration:</u>

$$\left.\begin{matrix}\log_2 36=x\Leftrightarrow 2^x=36\\ \log_3 36=y\Leftrightarrow 3^y=36\end{matrix}\right\}\Rightarrow\ 2^x\cdot 3^y=36^2=(2^2\cdot 3^2)^2$$
$$=2^4\cdot 3^4\underset{(*)}{\Rightarrow}\left\{\begin{matrix}x=4\\ y=4\end{matrix}\right.\Rightarrow\ \frac{1}{\log_2 36}+\frac{1}{\log_3 36}=\frac{1}{x}+\frac{1}{y}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

The proof in wrong in step (*). This step is only correct if beforehand x, y $\in\mathbb{N}$. Which is not true when, previously, we have written: $2^x = 36 = 2^2\cdot 3^2$

February 22-23: Let $a_1, a_2, \ldots, a_n$ not negative reals. We define M as the sum of all the products of pairs $a_i \cdot a_j$ $(i < j)$, that is:

$$M = a_1 \cdot (a_2 + a_3 + \cdots + a_n) + a_2 \cdot (a_3 + \cdots + a_n) + \cdots + a_{n-1} \cdot a_n$$

Prove that the square of any of the numbers $a_1, a_2, \ldots, a_n$ does not exceed $\frac{2M}{n \cdot (n-1)}$

Solution: By reduction to the absurd. Let's suppose:

$$\forall i, \;\; a_i^2 > \frac{2M}{n \cdot (n-1)}$$

Taking positive square roots (since the a_i are non-negative)

$$a_i > \sqrt{\frac{2M}{n \cdot (n-1)}}$$

Therefore:

$$\forall i, j \;\; (1 \le i < j \le n) \;\;\; a_i \cdot a_j > \frac{2M}{n \cdot (n-1)}$$

And then:

$$\begin{aligned}\mathbf{M} &= a_1 \left(\overbrace{a_2 + a_3 + \cdots + a_n}^{n-1} \right) + a_2 \left(\overbrace{a_3 + a_4 + \cdots + a_n}^{n-2} \right) + \cdots + a_{n-1} \cdot a_n \\ &> \frac{2M}{n \cdot (n-1)} (n-1) + \frac{2M}{n \cdot (n-1)} (n-2) + \cdots + \frac{2M}{n \cdot (n-1)} \cdot 1 \\ &= \frac{2M}{n \cdot (n-1)} \big((n-1) + (n-2) + \cdots + 1 \big) = \frac{2M}{n \cdot (n-1)} \frac{n-1+1}{2} (n-1) = \mathbf{M}\end{aligned}$$

That is, M > M, which is absurd. Later:

$$\exists i \in \{1, 2, \cdots, n\} \mid \; a_i^2 \le \frac{2M}{n \cdot (n-1)}$$

February 24: A 3x3 grid is filled with positive numbers so that the product of the numbers in each row and each column is 2 and the product of the 4 numbers in each of the four 2x2 grids is 4. What is the number in the center square of the grid?

Solution: We will have:

A	B	C	2
D	E	F	2
G	H	I	2
2	2	2	

Multiplying the four numbers of all the four 2x2 grids we have:

$$(2^2)^4 = (\mathbf{A \cdot B \cdot D \cdot E}) \cdot (\mathbf{B \cdot C \cdot E \cdot F}) \cdot (\mathbf{D \cdot E \cdot G \cdot H}) \cdot (\mathbf{E \cdot F \cdot H \cdot I})$$

Regrouping factors so that row and column products appear:

$$\begin{aligned} 2^8 &= (\mathbf{A \cdot B \cdot D \cdot E}) \cdot (\mathbf{B \cdot C \cdot E \cdot F}) \cdot (\mathbf{D \cdot E \cdot G \cdot H}) \cdot (\mathbf{E \cdot F \cdot H \cdot I}) \\ &= (\mathbf{A \cdot B \cdot C}) \cdot (\mathbf{D \cdot E \cdot F}) \cdot (\mathbf{B \cdot E \cdot H}) \cdot (\mathbf{D \cdot E \cdot F}) \cdot (\mathbf{G \cdot H \cdot I}) \cdot \mathbf{E} = \mathbf{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot E} \\ &= \mathbf{2^5 \cdot E} \;\;\Rightarrow\;\; \mathbf{2^3 = E = 8}\end{aligned}$$

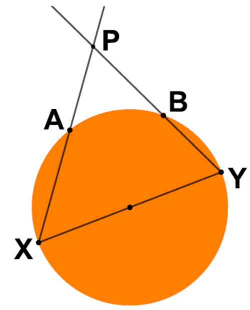
February 25: Prove that if p and p + 2 are both primes greater than 3, then 6 is a factor of p + 1.

Solution: We have three consecutive positive integers greater than three, with p and p + 2 being primes (twin primes). We have to prove that p + 1 is a multiple of 6, that is, of 2 and 3.

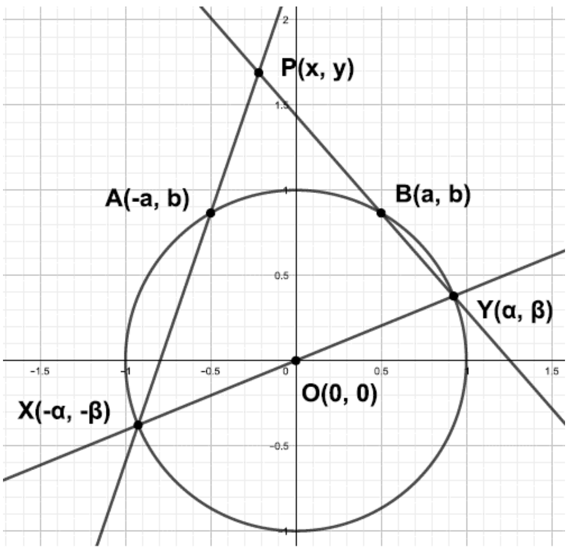
Since p is a prime greater than 3 it is not even. And since, between two consecutive integers, one is a multiple of 2, we will have that p + 1 is a multiple of 2.

Since p is greater than 3 and a prime, p is not a multiple of 3. That is, it must be true that p = 1 (3) or p = 2 (3). But if p = 1 (3) then it should be p + 2 = 0 (3) (different from 3) against p + 2 being prime. Then the only possibility that remains is p = 2 (3). And then p + 1 = 0 (3), which concludes the proof.

February 26-27: Let A and B be two non-diametrically opposite fixed points on a circle. Let X and Y be the ends of a diameter. Find the locus of the points P that are the intersection of the lines that pass through A and X and through B and Y



Solution: We choose a system of coordinate axes so that the center of the given circle is the origin of coordinates and that the Y axis is perpendicular to segment AB through its midpoint. With this choice we also make the radius of the circumference unity. Then:



$$0(0,0);\quad A(-a,b);\quad B(a,b);\quad X(-\alpha,-\beta);\quad Y(\alpha,\beta)$$

$$a^2+b^2=1=\alpha^2+\beta^2$$

Line that passes through A(− a, b) and X(− α, − β):

$$\frac{y-b}{x+a}=\frac{-\beta-b}{-\alpha+a};\quad (y-b)\cdot(\alpha-a)=(x+a)\cdot(\beta+b)$$

$$y(\alpha-a)=(x+a)\cdot(\beta+b)+b(\alpha-a)$$

and how $\alpha - a \neq 0$

$$y=\frac{\beta+b}{\alpha-a}\cdot(x+a)+b$$

Line that passes through B(a, b) and Y(α, β):

$$\frac{y-b}{x-a}=\frac{\beta-b}{\alpha-a};\quad (y-b)\cdot(\alpha-a)=(x-a)\cdot(\beta-b)$$

$$y(\alpha-a)=(x-a)\cdot(\beta-b)+b(\alpha-a)$$

and how $\alpha - a \neq 0$

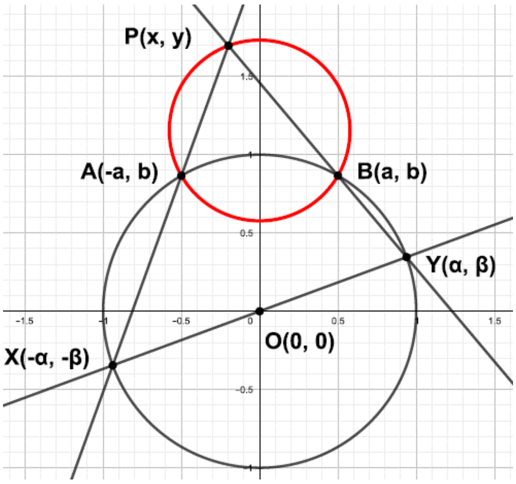
$$y=\frac{\beta-b}{\alpha-a}\cdot(x-a)+b$$

Cut-off point coordinates P(x , y):

$$\left.\begin{array}{l} y=\dfrac{\beta+b}{\alpha-a}\cdot(x+a)+b \\ y=\dfrac{\beta+b}{\alpha-a}\cdot(x+a)+b \end{array}\right\} \Rightarrow \frac{\beta+b}{\alpha-a}\cdot(x+a)+b=\frac{\beta+b}{\alpha-a}\cdot(x+a)+b \;\Rightarrow\; x=-\frac{\beta a}{b}\quad(1)$$

$$y=\frac{\beta+b}{\alpha-a}\cdot\frac{ab-\beta a}{b}+b=\frac{a\cdot(b^2-\beta^2)}{(\alpha-a)\cdot b}+b=\frac{b^2\alpha-a\beta^2}{b\cdot(\alpha-a)}\quad(2)$$

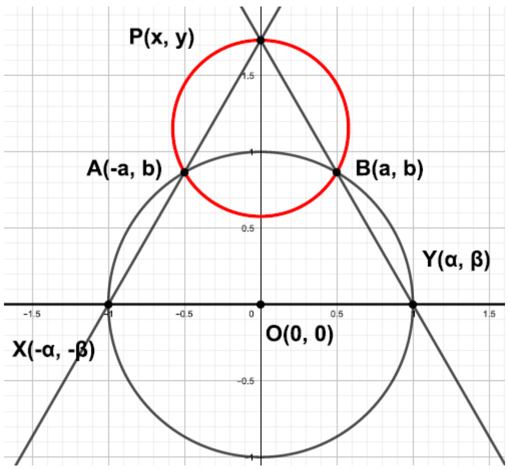
The coordinates of point P do not seem to suggest the equation of the locus. However, if we mentally translate point Y to the left, point P seems to describe the same type of equation as point Y, that is, point P (by translating point Y by the circumference) seems to describe another circle with center at the Y axis. This fact is also supported by what was obtained through the geogebra program

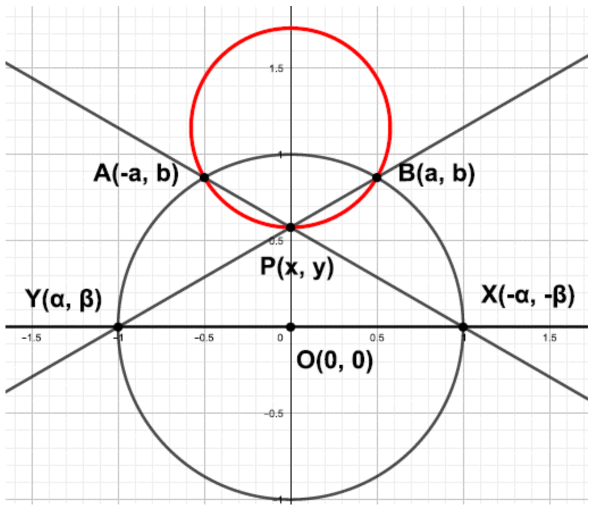


We calculate the presumed center by finding the midpoint of the points corresponding to $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (-1, 0)$, that is, when the point $Y(\alpha, \beta)$ is on the positive side of the X axis and when the point $Y(\alpha, \beta)$ is on the negative side of the X axis. Furthermore, the presumed radius will be half the length of the points P (x, y) obtained for these values from point Y

If we make in (1) and (2) $\alpha = 1$ and $\beta = 0$ we obtain that the coordinates of the cut-off point are

$$P\left(0, \frac{b}{1-a}\right)$$





If we make in (1) and (2) $\alpha = -1$ and $\beta = 0$ we obtain that the coordinates of the cut-off point are:

$$P\left(0, \frac{b}{1+a}\right)$$

Thus, the presumed center (0, A) and radius of the place should be:

$$(0, A) = PM\left(\left(0, \frac{b}{1-a}\right);\ \left(0, \frac{b}{1+a}\right)\right) = \left(\frac{0+0}{2}, \frac{\frac{b}{1-a} + \frac{b}{1+a}}{2}\right) = \left(0, \frac{1}{b}\right)$$

$$2r = \frac{b}{1-a} - \frac{b}{1+a} = \frac{2ab}{b^2} \quad \Rightarrow \quad r = \frac{a}{b}$$

To conclude that the locus is a circumference of center and radius, those obtained as presumable, we have to verify that the coordinates of the cut-off points P (x, y) comply with the equation of the circumference. Have:

$$\begin{aligned} x^2 + \left(y - \frac{1}{b}\right)^2 &= \frac{\beta^2 a^2}{b^2} + \left(\frac{b^2\alpha - a\beta^2}{b(\alpha - a)} - \frac{1}{b}\right)^2 = \frac{\beta^2 a^2}{b^2} + \left(\frac{b^2\alpha - a\beta^2 - \alpha + a}{b(\alpha - a)}\right)^2 \\ &= \frac{\beta^2 a^2}{b^2} + \left(\frac{(1-a^2)\alpha - a(1-\alpha^2) - \alpha + a}{b(\alpha - a)}\right)^2 = \frac{\beta^2 a^2}{b^2} + \left(\frac{a\alpha(\alpha - a)}{b(\alpha - a)}\right)^2 = \frac{\beta^2 a^2}{b^2} + \frac{\alpha^2 a^2}{b^2} \\ &= \frac{a^2(\beta^2 + \alpha^2)}{b^2} = \frac{a^2}{b^2} \end{aligned}$$

With what we have that the geometric place of the points P (x, y) is the circumference of center$\left(0, \frac{1}{b}\right)$ and of radius $r = \frac{a}{b}$