SOLUTIONS MARCH 2021

PROBLEMS FOR THE PREPARATION OF THE OLYMPIC 1 AND 2 OF THE E.S.O. CONVENED BY THE FESPM IN 2003 AND 2004. ORGANIZATION: JOSÉ COLÓN LACALLE. RETIRED TEACHER

March 1-8: In a cross-country competition, three runners start from A for B at the same time. The fastest reaches B an hour before noon and its speed is 15 km / h. The slowest reaches B an hour after noon and its speed is 10 km / h. The other runner reaches B at exactly noon. Find out A's starting time, the distance between A and B and the speed of the second runner

Solution: Let d be the distance between A and B. We will assume that they leave at x hours from A:

$$\left.\begin{array}{l} v_A = 15 = \dfrac{d}{11 - x} \quad \Rightarrow \quad 165 - 15x = d \\ v_C = 10 = \dfrac{d}{13 - x} \quad \Rightarrow \quad 130 - 10x = d \end{array}\right\}$$

Subtracting both equations:

$$35 = 5x \quad \Rightarrow \quad x = 7\ \text{h}$$

Substituting in the first equation:

$$d = 165 - 15 \cdot 7 = 60\ \text{km}$$

Finally:

$$v_B = \frac{60\ \text{km}}{(12 - 7)\text{h}} = 12\ \text{km/h}$$

March 2-3: Dani, Aitana and Laia have to distribute 21 bottles of soda left over from a party: 7 are full, 7 are half empty and 7 are empty. How should the cans be distributed so that the 3 get the same number of bottles and the same amount of soda? (You cannot transfer soda from one bottle to another)

Solution: Each one has to take (21/3 =) 7 bottles and the same amount of liquid ((7 + 3.5) / 3 =) 3.5, the equivalent of 3 full bottles and one half full. We will have:

Dani	2 empty	2 full	3 half empty	7 bottles
Aitana	2 empty	2 full	3 half empty	7 bottles
Laia	3 empty	3 full	1 half empty	7 bottles
	7 empty	7 full	7 half empty	

March 4-11: In a fruit bowl there are apples, pears and oranges. Dani and Laia pick different fruits. Dani and Aitana pick the same fruit. Neither Dani nor Laia pick pears. If Laia takes an apple, Aitana also. What fruit did each pick?

Solution: From the statement, we will have:

(1) Dani and Laia different fruits.

(2) Dani and Aitana the same fruit.

(3) Neither Dani nor Laia pears.

(4) If Laia apple, then Aitana apple.

Condition (4) is in contradiction with (1) and (2). Then Laia doesn't pick apples. For (3) Laia does not pick pear either. Then Laia picks orange.

For (3) Dani does not pick pear. For (1) Dani doesn't pick orange either. Then Dani takes an apple.

Finally, for (2), Aitana takes an apple.

March 5-6: Dani must multiply by 78 a two-digit number in which the tens digit is three times greater than the ones digit. By mistake she interchanges the digits of this factor and obtains a number that is 2808 units less than the product she is looking for. What is this product?

Solution: Let a be the ones digit of the two-digit number in the sentence. We will have that the sought factor is $3a \cdot 10 + a$. The condition of the statement is equivalent to:

$(3a\cdot10 + a)\cdot78 = (a\cdot10 + 3a)\cdot78 + 2808 \Rightarrow 31\cdot a\cdot78 - 78\cdot13a = 2808 \Rightarrow 78\cdot(31a - 13a) = 2808 \Rightarrow 78\cdot18a = 2808$

$$a = \frac{2808}{78 \cdot 18} = 2$$

Then the factor by which he had to multiply 78 is $(3a \cdot 10 + a =)$ 62. The product sought is $(78 \cdot 62 =)$ 4836

March 9: In this product each letter represents a number and different letters indicate different numbers. What is the value of AGUA?

$$\begin{array}{cccc} G & O & T & A \\ & & \times & A \\ \hline A & G & U & A \end{array}$$

Solution: We will have that A^2 ends in A and from the table attached to the right, that $A \in \{0, 1, 5, 6\}$. Obviously A cannot be 0. If it were, the result of the multiplication would be 0 against the result being a four-digit number. Obviously A cannot be 1. If it were, the result of the multiplication will be equal to the first factor and then we would have G = A = O and T = U which contradicts the statement. If A = 5, we will have:

A	A^2
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
0	0

G O T 5 x 5 5 G U 5	$\Rightarrow 5\cdot G = 5 \Rightarrow G = 1$	1 O T 5 x 5 5 1 U 5	$\Rightarrow$ 5·O + (we carry) = 1 $\Rightarrow$ O = 0 and we carry 1
1 0 T 5 x 5 5 1 U 5	$\Rightarrow 5\cdot T + 2 < 20 \Rightarrow 3{,}6 > T \notin \{0, 1\} \Rightarrow T \in \{2, 3\}$ Si T = 2 $\Rightarrow$ U = 2 = T Absurd! Si T = 3 $\Rightarrow$ U = 7	1 0 3 5 x 5 5 1 7 5	

If A = 6, we will have:

G O T 6 x 6 6 G U 6	$\Rightarrow 6\cdot G = 6 \Rightarrow G = 1$	1 O T 6 x 6 6 1 U 6	$\Rightarrow$ 6·O + (we carry) = 1 $\Rightarrow$ O = 0 and we carry 1
1 0 T 6 x 6 6 1 U 6	$\Rightarrow 6\cdot T + 3 < 20 \Rightarrow 2{,}8 > T \notin \{0, 1\} \Rightarrow T = 2 \Rightarrow$ U = 5	1 0 2 6 x 6 6 1 5 6	

<u>March 10:</u> Find the 9 smallest consecutive naturals (of more than one digit), the first ending in 1, the second ending in 2, and so on, so that when dividing each of them by its last digit, the result always gives exact

<u>Solution:</u> Obviously {1, 2, 3, 4, 5, 6, 7, 8, 9} meets the requirement of the statement, but they are natural of one digit. We can rephrase the statement requirement by:

We look for the natural minor x ≠ 0, such that

$$10x + k = \dot{\hat{k}} \qquad \text{for } k \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Obviously, whatever x is, the above is true for k = 1.

Since $k = \dot{\hat{k}} \;\Rightarrow\; 10x = \dot{\hat{k}} - k = \dot{\hat{k}}$. Then we look for the natural minor x such that:

$$10x = \dot{\hat{k}} \text{ for } k = 2, 3, 4, 5, 6, 7, 8, 9.$$

Since 10 = 2·5, we look for the smallest x such that:

$$x = \dot{\hat{k}} \text{ para } k = 3, 4, 6, 7, 9$$

And since:

$$\left.\begin{array}{l} \text{if } x = \dot{\hat{9}} \;\Rightarrow\; x = \dot{\hat{3}} \\ \text{if } x = \dot{\hat{4}} \;\Rightarrow\; x = \dot{\hat{2}} \end{array}\right\} \;\Rightarrow\; x = \dot{\hat{6}}$$

we look for the smallest x such that:

$$x = \dot{\hat{k}} \text{ for } k = 4, 7, 9$$

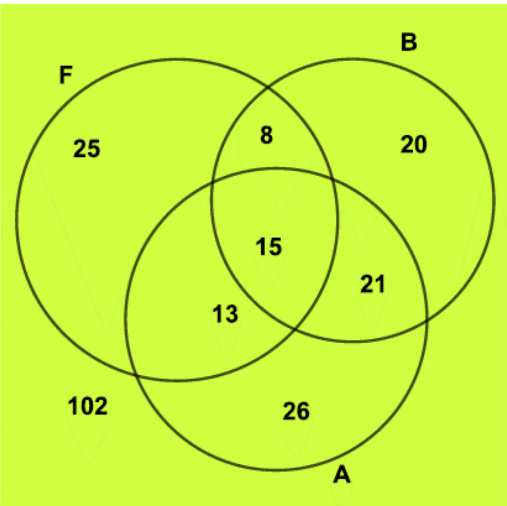
That is, x = lcm {4, 7, 9} = 9 · 4 · 7 = 252. The set of numbers in the statement is:

{2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529}

<u>March 12-13:</u> In a high school of 230 students, there are 15 who practice soccer, athletics and basketball, 23 who practice soccer and basketball, 36 who practice athletics and basketball, 28 who practice athletics and soccer, 61 who practice soccer, 64 who practice basketball and 75 who practice athletics. how many do not practice any sport?

<u>Solution:</u> We will represent by F (B, A) the group of students who practice soccer (basketball, athletics).

We will have the following Venn diagram (starting with the most complete intersection, that is, by F ∩ B ∩ A (of cardinality 15))



For example:

only F and B → 23 – 15 = 8

only F → 61 – (13 + 15 + 8) = 25

None of the three sports → 230 – (25 + 8 + 20 +15 + 13 + 21 + 26) = 102

Too:

$$\begin{aligned} |A \cup F \cup B| &= |A| + |F| + |B| - |A \cap F| - |A \cap B| \\ &\quad - |F \cap B| + |A \cap B \cap F| \\ &= 75 + 61 + 64 - 28 - 36 - 23 + 15 \\ &= 128 \end{aligned}$$

And, finally, if E is the set of all students of the high school.

$$|\overline{F} \cap \overline{A} \cap \overline{B}| = |\overline{F \cup A \cup B}| = |E| - |A \cup F \cup B| = 230 - 128 = 102$$

March 15-16: A freelancer manufactures a certain batch of parts in 12 days working 7 hours a day, of which he has dedicated 1 hour each day to preparing machines and material. How many more minutes must he work each day to complete the batch of parts in 10 days?

Solution: The worker works to manufacture the batch, a total of (12 · 6 =) 72 hours. If he must make the batch in ten days, he must work every day (72/10 =) 7.2 hours = 7 h 12’. As before he had to work 6 hours and now, he must work 7 hours and 12 minutes, he must increase the daily shift by (7 h 12’- 6 h =) 1h 12’

March 17-24: Three friends live on the same street, two of them in adjoining buildings. One of them tells the others: "The numbers we live in are prime and their product forms the last six digits of my mobile number." If the street has less than a hundred numbers, find the house numbers and the last numbers of the mobile

Solution: The numbers of the neighbouring houses differ by 2. Therefore, the neighbouring houses numbered with prime numbers must be twin primes. Twin prime numbers less than 100 are:

(3,5) (5, 7) (11, 13) (17, 19) (29, 31) (41,43) (59, 61) (71, 73)

We have to find another prime less than 100, which, multiplied by the numbers of each pair of twin primes, of a six-digit number.

Since:

$$111{,}23\ldots = \frac{100000}{29 \cdot 31} \leq x \leq \frac{999999}{29 \cdot 31} = 1112{,}34\ldots$$

The pair (29, 31) (and neither does any of the ones before her) work because their product does not give a six-digit number. We will begin to obtain solutions with the pair (41, 43).

Since:

$$56{,}72\ldots = \frac{100000}{41 \cdot 43} \leq x \leq \frac{999999}{41 \cdot 43} = 567{,}21\ldots$$

are solutions, 41, 43 and any prime between 57 and 100: 59, 61, 67, 71, 73, 79, 83, 89 and 97. With the following pair of twin primes, (59, 61), we will have:

$$27{,}78\ldots = \frac{100000}{59 \cdot 61} \leq x \leq \frac{999999}{59 \cdot 61} = 277{,}85\ldots$$

Therefore, 59, 61 and any prime between 28 and 100 are solutions: 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

With the last pair of twin cousins, (71, 73), we have:

$$19{,}29\ldots = \frac{100000}{71 \cdot 73} \leq x \leq \frac{999999}{71 \cdot 73} = 192{,}93$$

Therefore, 71, 73 and any prime between 20 and 100 are solutions: 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

There are a total of (9 + 16 + 17 =) 42 solutions to the problem. All of them are collected in the following Excel sheet

41	43	x	41·43·x		59	61	x	59·61·x		71	73	x	71·73·x
41	**43**	**59**	**104017**		**59**	**61**	**29**	**104371**		**71**	**73**	**23**	**119209**
41	**43**	**61**	**107543**		**59**	**61**	**31**	**111569**		**71**	**73**	**29**	**150307**
41	**43**	**67**	**118121**		**59**	**61**	**37**	**133163**		**71**	**73**	**31**	**160673**
41	**43**	**71**	**125173**		**59**	**61**	**41**	**147559**		**71**	**73**	**37**	**191771**
41	**43**	**73**	**128699**		**59**	**61**	**43**	**154757**		**71**	**73**	**41**	**212503**
41	**43**	**79**	**139277**		**59**	**61**	**47**	**169153**		**71**	**73**	**43**	**222869**
41	**43**	**83**	**146329**		**59**	**61**	**53**	**190747**		**71**	**73**	**47**	**243601**
41	**43**	**89**	**156907**		**59**	**61**	**59**	**212341**		**71**	**73**	**53**	**274699**
41	**43**	**97**	**171011**		**59**	**61**	**61**	**219539**		**71**	**73**	**59**	**305797**
					59	**61**	**67**	**241133**		**71**	**73**	**61**	**316163**
					59	**61**	**71**	**255529**		**71**	**73**	**67**	**347261**
					59	**61**	**73**	**262727**		**71**	**73**	**71**	**367993**
					59	**61**	**79**	**284321**		**71**	**73**	**73**	**378359**
					59	**61**	**83**	**298717**		**71**	**73**	**79**	**409457**
					59	**61**	**89**	**320311**		**71**	**73**	**83**	**430189**
					59	**61**	**97**	**349103**		**71**	**73**	**89**	**461287**
										71	**73**	**97**	**502751**

March 18-19: The bakery on my street sells the muffins for € 0.30 per unit. He also sells them in packs of 7 muffins at € 1 a pack and in packs of 12 muffins at € 1.80 a pack. My mother gave me a € 10 bill and told me to buy 60 muffins and keep the turns. How should I order to keep the most money?

Solution: Since each individual cupcake is worth € 0.3, each cupcake in a pack of 7 is worth (1/7 =) $0,1\widehat{42857}$€ and each cupcake in a pack of 12 is worth (1.80 / 12 =) 0 ,15 € it seems logical to buy as many packages of 7 as possible. As: 60 = 7 · 8 +4, it seems logical to buy 8 packages of 7 muffins and 4 individual ones with cost (8 · 1 + 4 · 0.30 =) € 9.20 which would provide us with a profit of (10 - 9.20 =) € 0.80. However, if we substitute a pack of 7 for one of 12, we have (7·7 + 12 =) 61 muffin, with a cost of (1·7 + 1.80 =) € 8.80 and a profit of (10 - 8.80 =) € 1.20 (and a cupcake). This is the optimal solution.

If instead of 4 individual muffins, we buy 9 packages of 7, we would have (9 · 7 =) 63 muffins with cost (9 · 1 =) 9 € and a profit of (10 - 9 =) 1 € (and 3 muffins), which is not bad as an optimal solution either.

March 20-27: As always different letters represent different digits. Deduct the values of the letters A, B, C and D so that the attached operation is correctly carried out.

	A	B	C
	x	C	A
	D	B	A
A	B	C	
A	D	C	A

Solution: First of all, let us note that C must be 1, since in the multiplication indicated when multiplying the first factor by the digit C, the first factor is repeated.

We will have then, that the multiplication remains

	A	B	1
	x	1	A
	D	B	A
A	B	1	
A	D	1	A

Now let's look at the sum highlighted in blue. Since B is a digit, B + 1 = 11 takes us to B = 10 (absurd!), Then B = 0. With which we will have that the multiplication remains as a figure on the right.

If we now look at the multiplication highlighted in green, we will have $A^2 = D$ and since each letter must be a different digit from 0 and 1, A can be 2 or 3, in which case d must be 4 or 9

	A	0	1
	x	1	A
	D	0	A
A	0	1	
A	D	1	A

There are two possible solutions:

A = 2; D = 4; C = 1 y B = 0

	2	0	1
	x	1	2
	4	0	2
2	0	1	
2	4	1	2

A = 3; D = 9; C = 1 y B = 0

	3	0	1
	x	1	3
	9	0	3
3	0	1	
3	9	1	3

<u>March 22-29:</u> Next to it is the leader board of a summer futsal tournament in which 4 teams participate: A, B, C and D. Each team played a match against the other three. The tournament had the peculiarity that there were no matches that ended with the same number of goals scored. Find out reasonably the results of all matches.

Matches: J (G, E, P) played (won, tied, lost). **Goals**: F (C) for (against)

	matches				goals	
	J	G	E	P	F	C
A	3	2	1	0	4	1
B	3	2	0	1	8	4
C	3	1	0	2	1	6
D	3	0	1	2	2	4

<u>Solution:</u> The table shows that A and D have tied a match. Then A and D have tied the match A-D. Since A has a goal against, that match has ended 0-0 or 1-1.

If A-D ends 0-0, A's other games must have ended 2-1 and 2-0 or 3-1 and 1-0.

If A-D ends 1-1 the other games played by A must have ended 1-0 and 2-0.

Suppose A-D ends at 0-0, then the other games played by D must have ended at 2-3 and 0-1 (*).

Suppose A-D ends at 1-1, then the other matches played by D must have ended at 1-2 and 0-1, but then we have one match ended at 0-1 (the one played by A) and another ended at 0- 1 (the one played by D) that contradicts the problem statement.

With this we have that the match A-D ends in 0-0.

If A-D ends with 0-0 and the other matches played by A end 3-1 and 1-0, we will have that there is a match played by A that ends with 1 goal and another match played by D that also ends with 1 goal (*) which contradicts the statement.

That is, we will have, with what has been reasoned so far:

OUTCOME	MATCH	NUMBER OF GOALS
0-0	A-D	0
2-1	A-(B o C)	3
2-0	A-(C o B)	2

2-3	D-B(**)	5
0-1	D-C	1

((**) It cannot be D-C because goals in favour of C count only one goal)

Suppose that 2-1 is the result A-B, then 2-0 is the result of A-C. Let's analyse the BC match, without this match we have that the goals for B are (1 + 3 =) 4 and that the goals against B are (2 + 2 =) 4, the BC match must have ended with 4-0.

That is, one possibility of outcomes is (***)

MATCH	OUTCOME
A-D	0-0
A-B	2-1
A-C	2-0
D-B	2-3
D-C	0-1
B-C	4-0

The other possibility

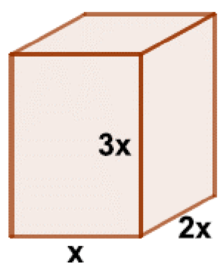
OUTCOME	MATCH	NUMBER OF GOALS
0-0	A-D	0
2-1	A- C	3
2-0	A-B	2
2-3	D-B	5
0-1	D-C	1

it is impossible, then, in the absence of party B-C, we have:

	F	C
B	3	4
C	2	2

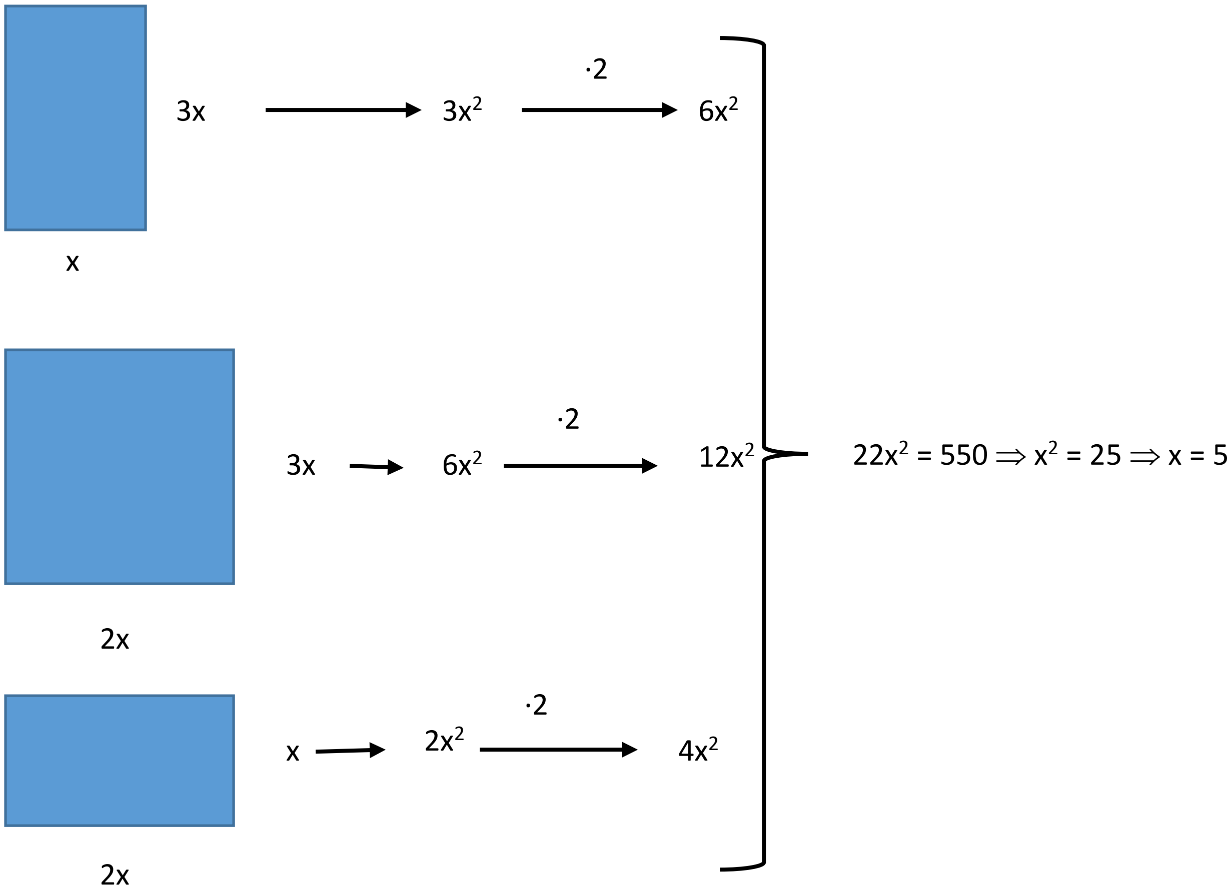
So, on the one hand, the B-C result must be 5-0, and on the other hand it must be 2-0. Therefore, the only possibility is that contemplated in (***)

March 23: The lengths of the sides of a straight prism with a rectangular base are proportional to 1, 2, and 3. The total surface area of the prism is 550 cm^2. Find the volume.



Solution: Let x, 2x and 3x be the dimensions of the right prism with a rectangular base. Its volume will be: $6x^3$. We must derive the value of x from knowing that its total area is 550 cm^2.

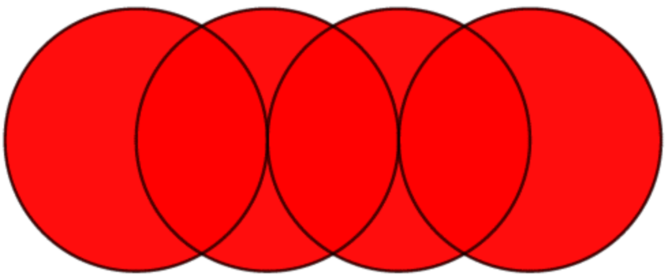
We will have:



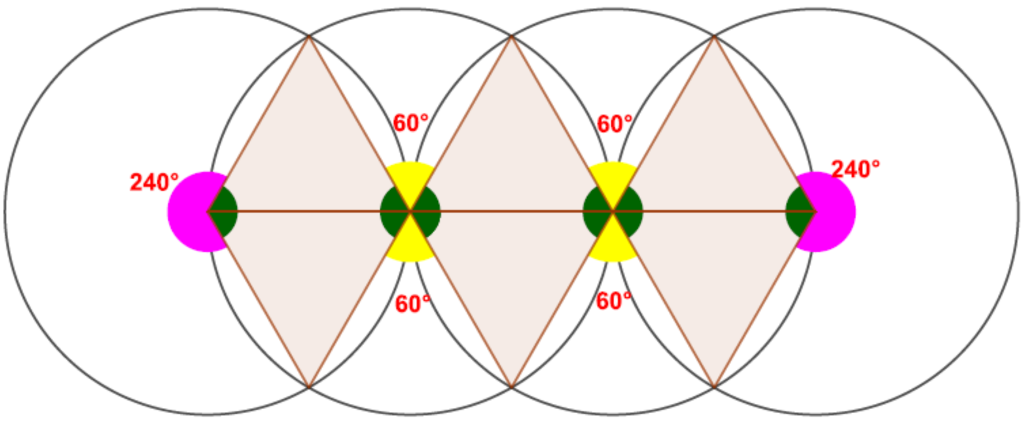
Therefore:

$$V = 6x^3 = 6 \cdot 5^3 = 750\ cm^3$$

March 25-26: Find the perimeter of the area generated by the four circles, each with a radius of 10 cm and in which each circle passes through the centers of the adjoining circles.



Solution: The triangles in the attached figure are equilateral, since each side measures (r =) 10 cm



Therefore, the green angles each measure 60°. Hence the yellow angles measure (180° - 2 · 60° =) 60° and the purple angles (360° - 2 · 60° =) 240°. Furthermore, the perimeter of the figure is the perimeter of two sectors of central angle 240° plus the perimeter of four circular sectors of central angle 60°. Since 2 · 240° + 4 · 60° = 720° = 2 · 360°, the perimeter of all these sectors corresponds to the perimeter of two circles. In short, the perimeter of the figure is:

$$2 \cdot 2\pi r = 4\pi \cdot 10 = 40\pi \text{ cm}$$

March 30-31: The 2nd year students of the E.S.O. They go camping and decide to leave a candle lit every night during the camp. Knowing that if the candle remains lit all night there is a quarter of a candle and therefore with the remains of the 4-night candles there is a candle that can be used another night, how many nights will they be camping if they buy 16 candles? Calculate the least number of candles they must buy if they want to have candlelight for 105 nights.

Solution: If we have 16 candles, we have for 16 nights and there will be 16 remaining candle quarters, which will generate (4/16 =) 4 more candles, which will cover 4 more nights and there will be 4 remaining candle quarters, which will generate (4/4 =) 1 more candle, which will cover one more night and there will be a quarter of a candle left over. In total with 16 candles we will have light for (16 + 4 + 1 =) 21 nights and there will be a quarter of a candle left over.

To answer the second question posed, we reason as follows

candles bought	Nights with light	Remains of candle
$4^2 = 16$	$4^2 + 4^1 + 4^0 = 21$	¼ candle
$4^3 = 64$	$4^3 + 4^2 + 4^1 + 4^0 = 85$	¼ candle
$4^3 + 4^2 = 64 + 16 = 80$	$4^3 + 2 \cdot (4^2 + 4^1 + 4^0) = 106$	½ candle

Then with 80 candles we have for 106 nights and half a candle left over. If we test with 79 candles, we have:

79 | 4
3 **19** | 4
3 **4** | 4
0 **1** | 4
1 **0**

79 + 19 + 4 + 1 + 0 = 103 nights

And now from the remains of the divisions and the remains of the last four candles and the one that comes from the remains of the divisions, we have:

$$\frac{3}{4} + \frac{3}{4} + \frac{1}{4} + \frac{1}{4} = 2$$

that reaches (103 + 2 =) 105 nights