SOLUTIONS APRIL 2021

PROBLEMS FOR BACHILLERATO (16-18 YEARS). ORGANIZATION: RAFAEL MARTÍNEZ CALAFAT. RETIRED PROFESSOR.

April 1-2: Is it possible to place addition or subtraction signs in empty cells so that the resulting expression is:

1. 11?
2. odd?
3. 10, at least ten different ways?

5		**6**		**7**		**8**		**9**		**10**		**11**		**12**		**13**		**14**		**15**

Solution: For 1, we have:

If we designate by A, the sum of the cells preceded by the + sign and by B the sum of the cells preceded by the – sign, we will have that if their sum is required to be 11:

$$\left.\begin{array}{r}\dfrac{5+15}{2}\cdot 11 = \mathrm{A}+\mathrm{B}\\ 11 = \mathrm{A}-\mathrm{B}\end{array}\right\} \Leftrightarrow \left.\begin{array}{r}10\cdot 11 = \mathrm{A}+\mathrm{B}\\ 11 = \mathrm{A}-\mathrm{B}\end{array}\right\}$$

Adding the two equations we arrive at: $11 \cdot 11 = 2A$, which is absurd since the first member is odd and the second member is even. So the answer to the first question is no.

For 2, we have:

If I is the odd number that we intend to get, we have:

$$\left.\begin{array}{r}10\cdot 11 = \mathrm{A}+\mathrm{B}\\ \mathrm{I} = \mathrm{A}-\mathrm{B}\end{array}\right\} \Leftrightarrow \text{(adding up)}\,10\cdot 11+\mathrm{I} = 2\mathrm{A}$$

The first member is (even + odd =) odd while the second member is even, which is a contradiction. Therefore, the answer to the second question is no.

For 3, we have:

$$\left.\begin{array}{r}10\cdot 11 = \mathrm{A}+\mathrm{B}\\ 10 = \mathrm{A}-\mathrm{B}\end{array}\right\} \Leftrightarrow \begin{array}{ccccc}\text{(adding up)} & 120 = 2\mathrm{A} & \Leftrightarrow & \mathrm{A} = 60\\ \text{(subtracting)} & 100 = 2\mathrm{B} & \Leftrightarrow & \mathrm{B} = 50\end{array}$$

That is: the sum of numbers preceded by + sign must be 60 and the sum of the other numbers (those preceded by – sign), must be 50. We have to investigate if there are at least ten different ways to choose numbers from {6, 7, 8, 9, 10, 11, 12, 13, 14, 15} to obtain the sum 55 (since the number 5 cannot be preceded by the sign –). If we choose 15 (and 5), to complete the sum up to 60, we can choose two of these three pairs: 6-14, 7-13, 8-12 and 9-11. Then we already have, in this way, that there is at least:

$$\mathrm{C}_2^4 = \binom{4}{2} = \frac{4\cdot 3}{20} = 6$$

different ways of obtaining sum 60. Other ways of obtaining sum 60, not previously contemplated are:

choose: 5, 6, 7, 8, 9, 12, 13

choose: 5, 7, 9, 12, 13, 14

choose: 5, 6, 9, 12, 13, 15

choose: 5, 6, 10, 11, 13, 15

that together with the 6 at the beginning they get the ten required in the statement. Then the answer to part 3 is: yes.

April 3: Can a number with 200 zeros, 100 ones, 100 twos, and 100 threes be a perfect square?

Solution: A number is a perfect square if the exponents of the primes that appear in its factorial decomposition as a product of prime numbers is even. If the number is made up of 200 zeros, 100 ones, 100 twos, and 100 threes, the sum of their digits is:

$$200\cdot 0 + 100\cdot 1 + 100\cdot 2 + 100\cdot 3 = 600$$

Since 600 is divisible by 3, but not by 9, we will have that any of the numbers in the statement is divisible by 3, but not by 9. That is, any of the numbers in the statement has three in its factorial decomposition, but this, does not contain 3^2. Therefore, any of the numbers in the statement is not a perfect square.

April 5-6: Dani, using a spreadsheet, has created a collection of naturals, from 1 to 2021. From it, two are chosen from the collection and substituted by their sum. This process is repeated until only one number remains in the collection. What number is the one left?

Solution: What Dani does is, little by little, calculating the sum of the numbers written at the beginning, because each time he takes two and replaces them with their sum and repeats this process until he is left with a number. The final number will be:

$$\frac{2021+1}{2}\cdot 2021 = 2043231$$

April 7: Is it possible that adding an even number of fractions with unity numerators and odd denominators gives unity?

Solution: Suppose:

$$\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_{2m}+1}=1$$

Then:

$$\frac{\prod_{i\neq 1}(2a_i+1)+\prod_{i\neq 2}(2a_i+1)+\cdots+\prod_{i\neq 2m}(2a_i+1)}{(2a_1+1)\cdot(2a_2+1)\cdot\cdots\cdot(2a_{2m}+1)}=1$$

The denominator is the product (of an even number) of odd numbers and therefore it is odd.

Each addend of the numerator is the product of odd and therefore odd. Since there are an even number of addends, the total sum is even.

And we have already reached an absurdity: the numerator is even and the denominator is odd and its quotient can never give 1.

April 8: Is there a natural that is equal to ten times the product of its digits?

Solution: Let Ab = $a_1a \ldots a_nb$ the number expressed as an ordered collection of digits. The requirement of the statement is equivalent to:

$$10\cdot A + b = 10\cdot a_1\cdot a_2\cdot\cdots\cdot a_n\cdot b \qquad (1)$$

The left member of the equality (1) has the number of units b while the right member has the number of units 0. Then b = 0 and from here:

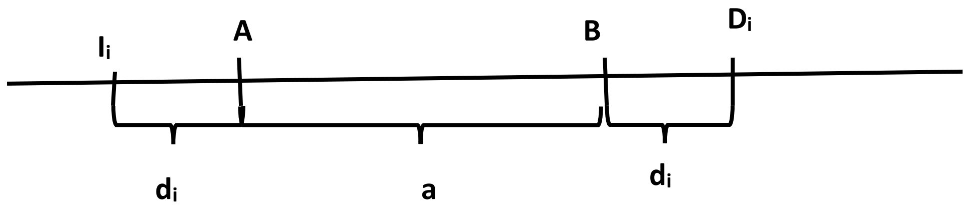
$$10 \cdot A + 0 = 10 \cdot a_1 \cdot a_2 \cdot \cdots \cdot a_n \cdot b = 10 \cdot a_1 \cdot a_2 \cdot \cdots \cdot a_n \cdot 0 = 0 \;\Rightarrow\; 10 \cdot A = 0 \;\Rightarrow\; A = 0$$

And, in short, that the starting number is 0. Therefore, there is no natural such that the number is equal to 10 times the product of the digits of the number.

<u>April 9-10:</u> Let a segment AB be given

a) Is it possible to choose an even number of points outside the segment AB, but on the line AB so that the sum of the distances from those points to A is equal to the sum of the distances from those points to B?
b) Is it possible to choose an odd number of points outside the segment AB, but on the line AB so that the sum of the distances from those points to A is equal to the sum of the distances from those points to B?

<u>Solution:</u> For the first question we have:



Given the segment (and line AB) and any even number 2n, we choose half of the points to the left of A and the other half of the points to the right of B such that:

$$d(I_i, A) = d_i = d(D_i, B) \quad \forall i \in \{1, 2, \cdots, n\}$$

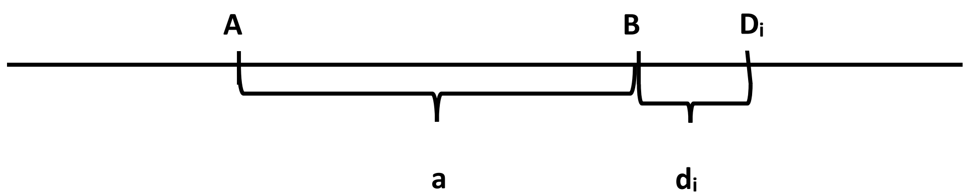
Then:

$$\sum_{i=1}^{n} \big(d(I_i, A) + d(D_i, A)\big) = \sum_{i=1}^{n} \big(d_i + (a + d_i)\big) = na + 2\sum_{i=1}^{n} d_i$$

$$\sum_{i=1}^{n} \big(d(I_i, B) + d(D_i, B)\big) = \sum_{i=1}^{n} \big((a + d_i) + d_i\big) = na + 2\sum_{i=1}^{n} d_i$$

And, therefore, the answer to a) is affirmative.

For question b), we have: The points must not necessarily be all to the left of A or to the right of B. If, for example, they are all to the right of B we will have

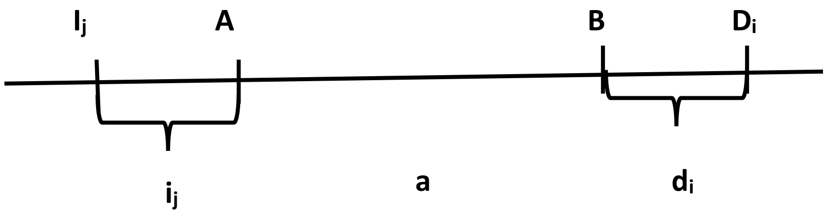


$$\sum_{i=1}^{2n+1} d(A, D_i) = \sum_{i=1}^{2n+1} (a + d_i) = (2n + 1)a + \sum_{i=1}^{2n+1} d_i \; ; \qquad \sum_{i=1}^{2n+1} d(B, D_i) = \sum_{i=1}^{2n+1} d_i$$

If:

$$\sum_{i=1}^{2n+1} d(A, D_i) = \sum_{i=1}^{2n+1} d(B, D_i) \quad \Rightarrow \quad (2n + 1)a = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad A = B$$

Therefore, some points must be to the right of B and others to the left of A. Since an odd number of points must be selected, to the right of B or to the left of A there must be an odd number (and in the other site an even number). Suppose that to the left of A there are an odd number of points and to the right of B there are an even number of points:



$$\forall j \in J \text{ with } |J| \text{ odd}, \quad \forall i \in I = \{1,2,3,\cdots,2n+1\} - J \quad \text{with } |I| \text{even}$$

Then:

$$D_1 = \sum_{j\in J} d(I_j, A) + \sum_{i\in I} d(D_i, A) = \sum_{j\in J} i_j + \sum_{i\in I} (a + d_i) = |I| \cdot a + \sum_{j\in J} i_j + \sum_{i\in I} d_i$$

$$D_2 = \sum_{j\in J} d(I_j, B) + \sum_{i\in I} d(D_i, B) = \sum_{j\in J} (a + i_j) + \sum_{i\in I} d_i = |J| \cdot a + \sum_{j\in J} i_j + \sum_{i\in I} d_i$$

If

$$D_1 = D_2 \quad \Rightarrow \quad (|I| - |J|)a = 0 \quad \Rightarrow \quad |I| = |J|$$

Which is absurd, since we are assuming that I is of even cardinality and J is of odd cardinality. Therefore, the answer to b) is negative.

<u>April 12:</u> If A is not a multiple of 3 and B is not a multiple of 3, can A·B be a multiple of 3?

<u>Solution:</u> Obviously the answer is negative.

$$\left.\begin{array}{l}\text{If A is not a multiple of 3} \Rightarrow \text{In A is not the factor 3}\\ \text{If B is not a multiple of 3} \Rightarrow \text{In B is not the factor 3}\end{array}\right\} \Rightarrow \text{in AB there is no factor } \Rightarrow \text{AB is not a multiple of 3}$$

Or alternatively:

$$\text{If A is not a multiple of 3} \Rightarrow \begin{cases} A = 1(3) \\ A = 2(3) \end{cases}, \quad \text{If B is not a multiple of 3} \Rightarrow \begin{cases} B = 1(3) \\ B = 2(3) \end{cases}$$

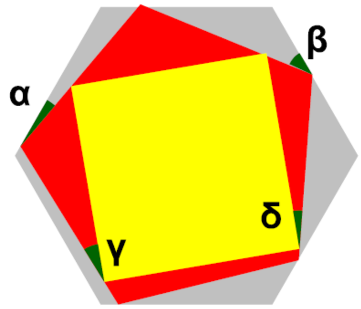
And the multiplication table provides us:

B\A	1(3)	2(3)
1(3)	1(3)	2(3)
2(3)	2(3)	1(3)

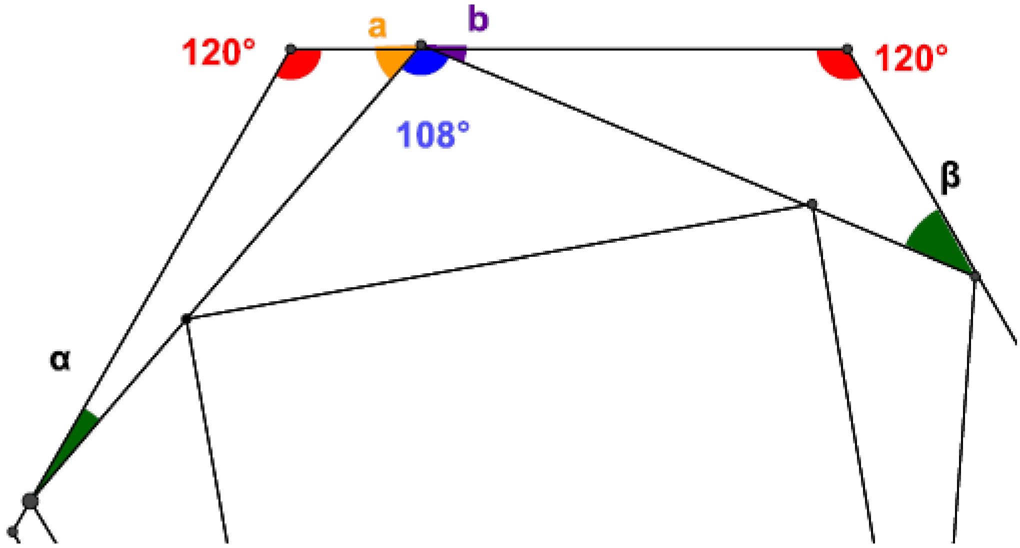
That is, AB = 1(3) or AB = 2(3). Therefore, AB is not a multiple of 3.

April 13-14: There is a square whose vertices touch the sides of a regular pentagon. In turn, the vertices of the pentagon are touching the sides of a regular hexagon. Calculate:

(1) $\alpha + \beta$
(2) $\gamma + \delta$



Solution: For the first question we will have:



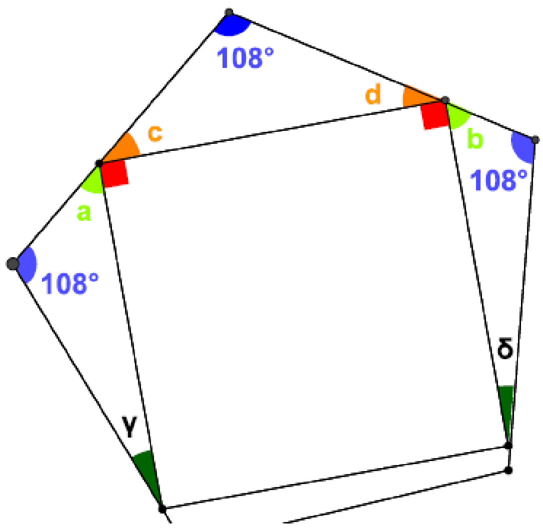
The angle between two consecutive edges of a hexagon is (central angle: 360/6 = 60° $\Rightarrow$ 180° – 60° =) 120°.
The angle between two consecutive edges of a pentagon is (central angle: 360/5 = 72° $\Rightarrow$ 180° – 72° =) 108°.

In the triangle on the left: $\alpha + 120° + a = 180° \Rightarrow a = 180° - (120° + \alpha)$

In the right triangle: $\beta + 120° + b = 180° \Rightarrow b = 180° - (120° + \beta)$.

Finally: $180° = a + b + 108° \Rightarrow 180° - (120° + \alpha) + 180° - (120° + \beta) + 108° = 180° \Rightarrow \alpha + \beta = 48°$

For the second question, we will have:



The angle between two consecutive edges of a pentagon is (central angle: 360/5 = 72° $\Rightarrow$ 180° – 72° =) 108°.
In the triangle on the left: $a = 180° - 108° - \gamma$
In the right triangle: $b = 180° - 108° - \delta$
What's more:

$$c = 180° - 90° - a = 18° + \gamma$$
$$d = 180° - 90° - b = 18° + \delta$$

And finally, in the central triangle:

$$108° + c + d = 180° = 108° + 18° + \gamma + 18° + \delta.$$

Where from:

$$\gamma + \delta = 36°$$

April 15: Prove that a natural has an odd number of divisors $\Leftrightarrow$ is a perfect square.

Solution: Remember that

$$\text{if } N = \prod_{i=1}^{n} p_i^{\alpha_i} \text{ then the number of divisors of N is } \prod_{i=1}^{n} (\alpha_i + 1)$$

$\Leftarrow$ If N is a perfect square, each prime of its factorial decomposition as a product of prime numbers has an even exponent. Therefore each factor $(\alpha_i + 1)$ is odd, and then:

$$\prod_{i=1}^{n} (\alpha_i + 1)$$

is a product of odd and therefore odd.

$\Rightarrow$ If a number has an odd number of divisors, then

$$\prod_{i=1}^{n} (\alpha_i + 1) \text{ it's odd} \Rightarrow (\alpha_i + 1) \text{ it's odd } \forall i \in \{1, 2, \cdots, n\} \Rightarrow \alpha_i \text{ it's even } \forall i \in \{1, 2, \cdots, n\}$$

(since if α_j were odd, then $\alpha_j + 1$ would be even and the product would be even). From what we deduce:

$$N = \prod_{i=1}^{n} p_i^{\alpha_i} = \left(\prod_{i=1}^{n} p_i^{\alpha_i/2} \right)^2 = (n)^2$$

That is, N is a perfect square.

April 16: A racing team has a car with which to travel 3,500 km for the practice of the season. Each day you can travel 300 or 500 or 700 km, can you complete the training plan in an even number of days?

Solution: Let x be the 300 km sessions, and the 500 km sessions and k –(x + y) the 700 km sessions. Must be fulfilled:

$$300x + 500y + 700 \cdot (k - x - y) = 3500; \quad 3x + 5y + 7 \cdot (k - x - y) = 35; \quad 7k - 35 = 4x + 2y$$

The right member is an even number, while the left member is, if k is even, (even – odd =) odd. Then the training plan cannot be carried out in an even number of days.

April 17: Find the naturals a such that **a + 2a + 3a + ⋯ + 9a** turns out to be a natural with all its figures equal.

Solution:

(1) **a + 2a + 3a + ⋯ + 9a = 45·a = k·11** $\Rightarrow 5 \cdot 3^2 \cdot a = k \cdot 11 \Rightarrow$ k to contain the factors 5 and 3^2, against being a digit.

(2) **a + 2a + 3a + ⋯ + 9a = 45·a = k·111** $\Rightarrow 5 \cdot 3^2 \cdot a = k \cdot 3 \cdot 37 \Rightarrow 5 \cdot 3 \cdot a = k \cdot 37 \Rightarrow$ k to contain the factors 5 and 3, against being a digit.

(3) **a + 2a + 3a + ⋯ + 9a = 45·a = k·1111** $\Rightarrow 5 \cdot 3^2 \cdot a = k \cdot 11 \cdot 101 \Rightarrow$ k to contain the factors 5 and 3^2, against being a digit.

(4) **a + 2a + 3a + ⋯ + 9a = 45·a = k·11111** $\Rightarrow 5 \cdot 3^2 \cdot a = k \cdot 41 \cdot 271 \Rightarrow$ k to contain the factors 5 and 3^2, against being a digit

(5) **a + 2a + 3a + ⋯ + 9a = 45·a = k·111111** $\Rightarrow 5 \cdot 3^2 \cdot a = k \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \Rightarrow 3 \cdot 5 \cdot a = k \cdot 7 \cdot 11 \cdot 13 \cdot 37 \Rightarrow$ k to contain the factors 5 and 3, against being a digit.

(6) **a + 2a + 3a + ⋯ + 9a = 45·a = k·1111111** $\Rightarrow 5 \cdot 3^2 \cdot a = k \cdot 239 \cdot 4649 \Rightarrow$ k to contain the factors 5 and 3^2, against being a digit.

(7) **a + 2a + 3a + ⋯ + 9a = 45·a = k·11111111** ⇒ $5\cdot3^2\cdot a = k\cdot11\cdot73\cdot101\cdot137$ ⇒ k to contain the factors 5 and 3^2, against being a digit.

(8) **a + 2a + 3a + ⋯ + 9a = 45·a = k·111111111** ⇒ $5\cdot3^2\cdot a = k\cdot3^2\cdot37\cdot333667$ ⇒ $5\cdot a = k\cdot37\cdot333667$

Equality holds if k = 5 and a = 37·3333667 = 12345679.

Once we have seen how the requirement works for the simplest cases, we return to the equation that originated.

$$45\cdot a = k\cdot\overbrace{1111\cdots11}^{m} \;\Leftrightarrow\; 3^2\cdot5\cdot a = k\cdot\overbrace{111\cdots\cdot11}^{m}$$

Since k has a single digit, k can only take the values 5, 3 or 9.

If k = 3 or k = 9, we have:

$$3\cdot5\cdot a = \overbrace{111\cdots11}^{m} \quad\Rightarrow\quad \overbrace{111\cdots\cdot11}^{m} = \dot{\hat{5}} \;\text{ Absurd}$$
$$5\cdot a = \overbrace{111\cdots11}^{m} \quad\Rightarrow\quad \overbrace{111\cdots\cdot11}^{m} = \dot{\hat{5}} \;\text{ Absurd}$$

If k = 5, then:

$$9\cdot a = \overbrace{111\cdots11}^{m} \quad\Rightarrow\quad \overbrace{111\cdots\cdot11}^{m} = \dot{\hat{9}} \;\Rightarrow \text{the number of ones is } m = 9n\ (9, 18, 27, 36, \ldots\ldots)$$

We already have a solution found for 9 ones: a_1 = 12345679. For 18 digits one, we will have:

$$5\cdot\overbrace{111\cdots11}^{18} = 5\cdot\left(\overbrace{111\cdots11}^{9}\cdot10^9 + \overbrace{111\cdots11}^{9}\right) = 45\cdot a_1\cdot10^9 + 45\cdot a_1 = 45\cdot(a_1\cdot10^9 + a_1)$$
$$= 45\cdot a_2$$

Namely: a_2 = 12345679012345679.

By induction, the solutions are:

$$a_{n+1} = a_n + 10^{9n}\cdot a_1$$
$$a_1 = 12345679$$

(The solution for 9n ones is preceded by a zero and all the ordered figures of the first solution). For n = 1 and n = 2 we already have it proven. Let's assume it to be true for n and let's see it for n + 1

$$45(a_n + 10^{9n}\cdot a_1) = 45a_n + 45a_1\cdot10^{9n} = 5\cdot\overbrace{111\cdots11}^{9n} + 5\cdot\overbrace{111\cdots11}^{9}\cdot10^{9n}$$
$$= 5\cdot\left(\overbrace{111\cdots11}^{9n} + \overbrace{111\cdots11}^{9}\cdot10^{9n}\right) = 5\cdot\overbrace{111\cdots11}^{9(n+1)}$$

Abril 19: Find the smallest natural number such that the number is equal to 5 times the product of its digits.

Solution: Obviously the numbers searched cannot have a single digit. Suppose the numbers have two digits: N = 10a + b. The requirement of the statement becomes

$$10a + b = 5ab \;\Rightarrow\; b = 5a(b-2) \quad (*)$$

Since 5a (b - 2) is a multiple of 5, b must be. Therefore, b = 0 or b = 5. Then

$$\text{if } b = 0 \;\Rightarrow\; 10a = 10a + b = 5ab = 0 \;\Rightarrow\; a = 0 \;\Rightarrow\; 10a + b = 0 \notin \mathbb{N}$$
$$\text{if } b = 5 \text{ in } (*) \text{ we have: } 5 = 5a(5-2) \;\Rightarrow\; 5 = 3\cdot5a \;\Rightarrow 1 = 3a \text{ absurd!}$$

Therefore, there is also no two-digit number that meets what is required.

Let's see what happens if the number has three digits: N = 100a + 10b + c. We will have:

$$N = 100a + 10b + c = 5 \cdot a \cdot b \cdot c \;\Rightarrow\; 100a + c = 5 \cdot a \cdot b \cdot c - 10 \cdot b = 5b(a \cdot c - 2)$$

Since 5b (ac - 2) is a multiple of 5, 100a + c must be. Therefore, c = 0 or c = 5.

If c = 0, then:

$$N = 100a + 10b + 0 = 5 \cdot a \cdot b \cdot 0 = 0 \notin \mathbb{N}$$

If c = 5, then:

$$100a + 5 = 5b(5a - 2) \;\Rightarrow\; 20a + 1 = b(5a - 2) \xrightarrow{5a-2\neq 0} b = \frac{20a + 1}{5a - 2} = 4 + \frac{9}{5a - 2}$$

Therefore 5a - 2 is a divisor of 9, that is, 5a - 2 must be ± 1, ± 3 o ± 9.

$$\text{if } 5a - 2 = \pm 1 \;\Rightarrow\; a \notin \mathbb{N}$$
$$\text{if } 5a - 2 = \pm 9 \;\Rightarrow\; a \notin \mathbb{N}$$
$$\text{if } 5a - 2 = -3 \;\Rightarrow\; a \notin \mathbb{N}$$
$$\text{if } 5a - 2 = 3 \;\Rightarrow\; a = 1, b = 4 + \frac{9}{3} = 7$$

And since, c = 5, we have that a number that meets the requirement in the statement is 175.

April 20: What is the largest integer m such that m always divides a:

$$\mathbf{n^2 \cdot (n^2 - 1) \cdot (n^2 - 4)}$$

for any integer n > 2?

Solution: Let's investigate possible values for m, looking at the divisors of $n^2 \cdot (n^2 - 1) \cdot (n^2 - 4) = (n - 2) \cdot (n - 1) \cdot n^2 \cdot (n + 1) \cdot (n + 2)$ for the first values of n

	n – 2	n – 1	n^2	n + 1	n+ 2		
n = 3	1	2	3^2	4	5	→	$1 \cdot 2^3 \cdot 3^2 \cdot 5$
n = 4	2	3	2^4	5	6	→	$2^4 \cdot 3^2 \cdot 5$
n = 5	3	4	5^2	6	7	→	$2^3 \cdot 3^2 \cdot 5^2 \cdot 7$
n = 6	4	5	6^2	7	8	→	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
n = 7	5	6	7^2	8	9	→	$2^4 \cdot 3^3 \cdot 5 \cdot 7^2$
:	:	:	:	:	:	:	:

It seems that $m = 2^3 \cdot 3^2 \cdot 5$ (a larger value of m would make m not divide a $n^2 \cdot (n^2 - 1) \cdot (n^2 - 4)$ for example for n = 3). It remains to be shown that m divides $n^2 \cdot (n^2 - 1) \cdot (n^2 - 4)$ for all values of n after n = 7. We will have:

$$n^2 \cdot (n^2 - 1) \cdot (n^2 - 4) = (n - 2) \cdot (n - 1) \cdot n^2 \cdot (n + 1) \cdot (n + 2) \quad (*)$$

As between two consecutive naturals there is one that is a multiple of two, in the expression (*) there are at least two multiples of 2, as well as those multiples of two are consecutive, one is a multiple of four. Therefore, in the expression (*) there is at least the factor 2^3. As among three consecutive naturals there is one that is a multiple of three, in the expression (*) there are at least two multiples of three:

$$\text{if } n - 2 = \dot{\hat{3}} \;\Rightarrow\; n + 1 = \dot{\hat{3}} \;\Rightarrow\; \text{in } (*) \text{ there are two factors } 3$$
$$\text{if } n - 1 = \dot{\hat{3}} \;\Rightarrow\; n + 2 = \dot{\hat{3}} \;\Rightarrow\; \text{in } (*) \text{ there are two factors } 3$$
$$\text{if } n = \dot{\hat{3}} \;\Rightarrow\; \text{in } (*) \text{ there are two factors } 3$$

Then the expression (*) is divisible by 3^2. As among five consecutive naturals there is at least one that is a multiple of five, we will have that in the expression (*) there is at least one factor five.

In short, in the expression (*) there are at least three factors 2, two factors 3 and a factor 5. Then the expression (*) is divisible by ($2^3 \cdot 3^2 \cdot 5 =$) 360, whatever n > 2.

April 21: What is the largest (smallest) eight-digit number that is divisible by 11?

Solution: The largest number of eight different digits is: 98765432. Since 2 - 3 + 4 - 5 + 6 - 7 + 8 - 9 = - 4, 98765432 - (- 4) = 98765436 is a multiple of 11. And, successively subtracting 11 from this number (until we achieve a number with all its different figures) we get multiples of 11:

$$98765436 \underset{-11}{\longrightarrow} 98765425 \underset{-11}{\longrightarrow} 98765414 \underset{-11}{\longrightarrow} 98765403$$

The greatest number of eight different digits multiple of 11 is 98765403.

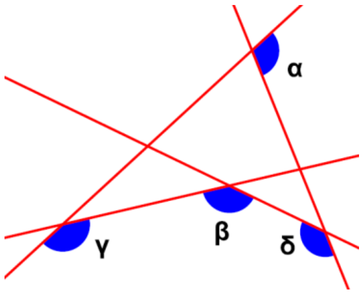
For the natural minor of eight different digits multiple of 11 we have:

The smallest number of eight different digits is: 10234567. Since 7 - 6 + 5 - 4 + 3 - 2 + 0 - 1 = 2, 10234567 - 2 = 10234565 is a multiple of 11. And, successively adding 11 to this number (up to get a number with all the different digits) we get multiples of 11:

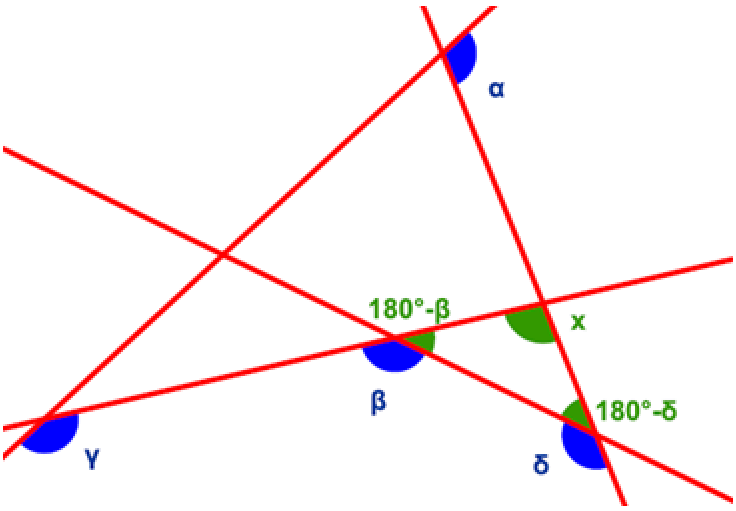
$$10234565 \underset{+11}{\longrightarrow} 10234576$$

The smallest number of eight different digits multiple of 11 is 10234576

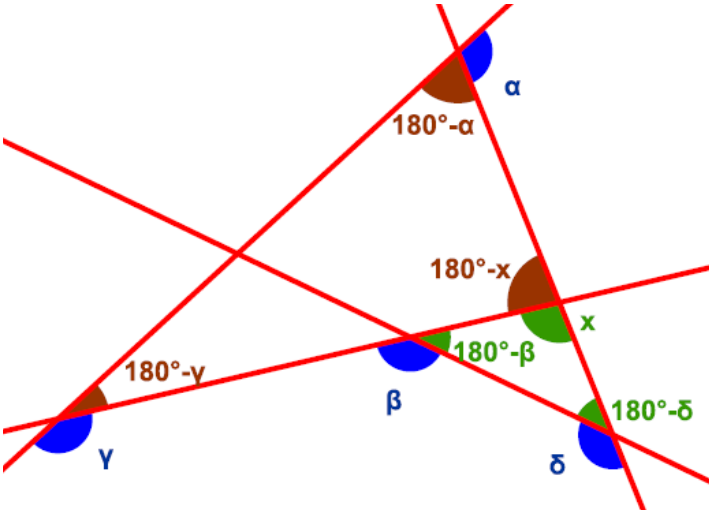
April 22: Calculate: $\alpha + \beta + \gamma + \delta$



Solution:



$$x + 180° - \beta + 180° - \delta = 180°$$
$$\Rightarrow \quad 180° + x = \beta + \delta \ (*)$$



$$180° - \alpha + 180° - x + 180° - \gamma = 180°$$
$$360° = \alpha + \gamma + x \quad (**)$$

Adding (*) and (**), we have:

$$180° + x + 360° = \beta + \delta + \alpha + \gamma + x \quad \Rightarrow \quad 540° = \beta + \delta + \alpha + \gamma$$

April 23-24: In a demographic investigation, a human group is classified according to certain criteria: sex (M, F) age (young, mature, retired) political tendency (extreme left, left, right, extreme right) and opinion of a political leader (0-2, 3-4, 5-6, 7-8, 9-10). In each class generated there are 10 people, that is, for example, there are 10 women, retired, right-wing who classify the political leader with 7-8 and so on. How many people are in the group? If a person is a woman, young, left-wing and with a score of 3-4, how many people are in the group who differ on exactly two criteria? And how many that differ on at least two criteria?

Solution: For the first question, let's see how many criteria there are

sex	age	tendency	opinion	total
2	3	4	5	(2·3·4·5 =) 120

If there are 10 people in each of them, the total number of people is (120·10 =) 1200.

For the second question we will have:

sex	age	tendency	opinion	number persons	total
female	young	left	3-4	1	1
female	young	3	4	10	(3·4·10 =) 120
female	2	left	4	10	(2·4·10 =) 80
female	2	3	3-4	10	(2·3·10 =) 60
1	young	left	4	10	(1·4·10 =) 40
1	young	3	3-4	10	(1·3·10 =) 30
1	2	left	3-4	10	(1·2·10 =) 20

For example, the third row would be the total number of people who match the one given in their sex (female) and age (young) and differ from it in trend and opinion and so on.

In total there are (120 + 80 + 60 + 40 + 30 + 20 =) 350 people who differ from the one considered in exactly two criteria.

For the third question, let's calculate how many people differ from the one considered only in one criterion

sex	age	tendency	opinion	number persons	total
female	young	left	3-4	1	1
female	young	left	4	10	(4·10 =) 40
female	young	3	3-4	10	(3·10 =) 30
female	2	left	3-4	10	(2·10 =) 20
1	young	left	3-4	10	(1·10 =) 10

There is (40 + 30 + 20 + 10 =) 100 people who differ from the one considered in exactly one criterion.

Finally: The number of people who differ from the one considered in at least two criteria is equal to the number of people who differ from the one considered minus the number of people who differ from the one considered in a single criterion. That is, there are ((1199 - 9) - 100 = 1190 - 100 =) 1090 people who differ from the one considered in at least two criteria.

April 26-27: Let us consider the sequences of positive integers in which, from the third term, the nth term is the arithmetic mean of the previous n-1 terms. How many of these sequences satisfy that $a_{100} = 1$?

Solution: Let $(a_n)_{n\in\mathbb{N}}$ be one of these sequences. So, we have:

$$a_1$$

$$a_2$$

$$a_3 = \frac{a_1 + a_2}{2}$$

$$a_4 = \frac{a_1 + a_2 + a_3}{3} = \frac{a_1 + a_2 + \frac{a_1 + a_2}{2}}{3} = \frac{a_1 + a_2}{2}$$

$$a_5 = \frac{a_1 + a_2 + a_3 + a_4}{4} = \frac{a_1 + a_2 + \frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2}}{4} = \frac{a_1 + a_2}{2}$$

$$a_6 = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = \frac{a_1 + a_2 + \frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2}}{5} = \frac{a_1 + a_2}{2}$$

It seems that the sequences we are talking about fulfill:

$$a_n = \frac{a_1 + a_2}{2} \quad \forall n \geq 3$$

Let's prove it by full induction. For n = 3 we have already tested it. Suppose that all terms up to n - 1 and after the second term coincide with the semisum of the first two and see that the nth term also fulfils it.

$$a_n = \frac{a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-2} + a_{n-1}}{n-1} = \left\{\begin{matrix} a_3 = \frac{a_1 + a_2}{2} \\ a_4 = \frac{a_1 + a_2}{2} \\ \vdots \\ a_{n-2} = \frac{a_1 + a_2}{2} \\ a_{n-1} = \frac{a_1 + a_2}{2} \end{matrix}\right\}$$

$$= \frac{(a_1 + a_2) + \overbrace{\frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2} + \cdots + \frac{a_1 + a_2}{2}}^{n-3}}{n-1} = \frac{\frac{n-1}{2}(a_1 + a_2)}{n-1} = \frac{a_1 + a_2}{2}$$

Then, the sequences we are talking about are those in which a1 and a2 are any natural and the subsequent terms are the arithmetic mean of the two initials. As $a_{100} = 1$

$$a_{100} = \frac{a_1 + a_2}{2} = 1 \quad (a_1, a_2 \in \mathbb{N} \Rightarrow)\ a_1 = a_2 = 1$$

Then, the only sequence of those considered that satisfies $a_{100} = 1$ is the sequence identically equal to 1

April 28: If $\mathbf{a^2 + b^2 = 3 \cdot a \cdot b}$ calculate:

$$\left(\frac{\mathbf{a+b}}{\mathbf{a-b}}\right)^{\mathbf{6}}$$

Solution:

$$\left(\frac{a+b}{a-b}\right)^6 = \left(\left(\frac{a+b}{a-b}\right)^2\right)^3 = \left(\frac{(a+b)^2}{(a-b)^2}\right)^3 = \left(\frac{a^2+b^2+2ab}{a^2+b^2-2ab}\right)^3 = \left(\frac{3ab+2ab}{3ab-2ab}\right)^3 = \left(\frac{5ab}{ab}\right)^3 = 5^3 = 125$$

April 29: Is it true that, if a natural is divisible by 28 and is divisible by 8, it is also divisible by (28 · 8 =) 224?

Solution 1: The answer to the question offered is no: If N is divisible by 28 (= $7 \cdot 2^2$), what can be said is that the factor 7 and factor 2^2 appear in the factorial decomposition of N. If, in addition, N is divisible by 8 (= 2^3) in the factorial decomposition of N, the factor 2^3 appears. Then in the factorial decomposition of N, the factor 7 and the factor 2^3 appear and the factor 7 and the factor 2^5 should not necessarily appear ($2^5 \cdot 7 = 28 \cdot 8 = 224$)

Solution 2: With a counterexample, enough.

$$\left.\begin{array}{l}56 \text{ is divisible by} 28\\ 56 \text{ is divisible by } 8\end{array}\right\}\ 56 \text{ is not divisible by } 224\ (= 28 \cdot 8)$$

April 30: Solve the equation:

1716·6!·7! = n!

Solution: We have

$$1716 = 2^2 \cdot 3 \cdot 11 \cdot 13 = 11 \cdot 12 \cdot 13$$

$$\left.\begin{array}{l}6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 8 \cdot 9 \cdot 10\\ 7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7\end{array}\right\} \Rightarrow \quad 6! \cdot 7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 10!$$

$$1716 \cdot 6! \cdot 7! = 10! \cdot 11 \cdot 12 \cdot 13 = 13!$$

Then n = 13