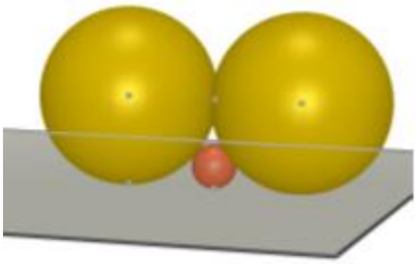
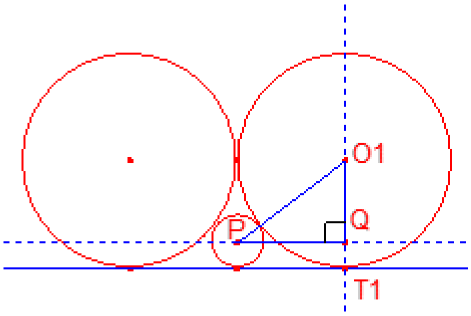
# SOLUTIONS JUNE 2021

PROBLEMS FOR USING GEOMETRIC PROGRAMS. AUTHOR: RICARD PEIRÓ I ESTRUCH. IES “Abastos”. Valencia.



**June 1-2:** You have two tangent and equal spheres on a table. What is the radius of the largest sphere that can pass between the two spheres above the table?

*Sangaku, Temple Kon'noh Hachiman, Tokyo. 1846*




**Solution:** Let r be the radius of the maximum sphere. This sphere will be tangent to the spheres of radius R.

The centers of the three spheres are in the same plane. We consider the section formed by the plane that passes through the centers of the two spheres of radius R and is perpendicular to the table.

Let O1 be the center of the sphere on the right. Let P be the center of the small sphere. Let T1 be the point of tangency of the sphere with center O1 and the table. Let Q be the projection of P on the line O1T1

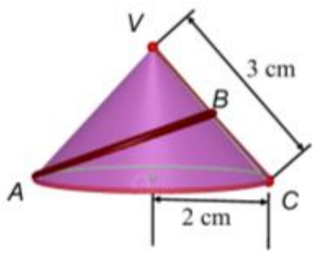
$$\overline{PQ} = R, \qquad \overline{PO1} = R + r, \qquad \overline{QO1} = R - r$$

Applying the Pythagorean theorem to the right triangle$\overset{\Delta}{PQO1}$

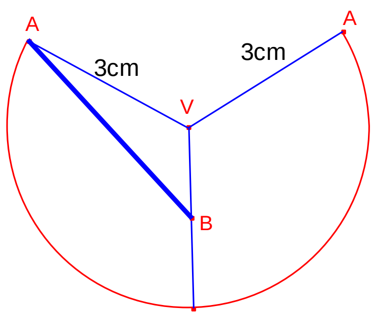
$$(R + r)^2 = (R - r)^2 + R^2$$

Simplifying:

$$4Rr = R^2; \quad r = \frac{1}{4}R$$




**June 3-4:** Let the solid cone have diameter AC = 4 cm, vertex V and generatrix AV = 3 cm. Let B be the midpoint of the generatrix CV. What is the minimum distance between A and B?




**Solution:** If we cut and develop the cone along the AV line, we have the attached figure. The arc of the sector is equal to the length of the circumference of radius 2 cm.

$$L_{arc} = 2\pi \cdot 2 = 4\pi$$

The central angle of the circular sector radius 3 cm is:

$$\alpha = \frac{4\pi}{3}\frac{180°}{\pi} = 240°$$

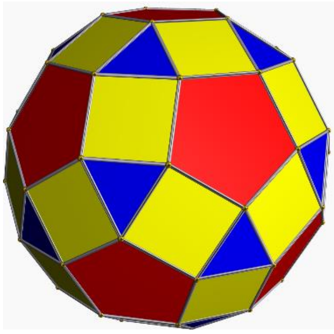
Point B is on the bisector of the anterior sector

$$\angle AVB = 120°$$

Applying the cosine theorem to the triangle $\overset{\Delta}{AVB}$

$$\overline{AB}^2 = 3^2 + \left(\frac{3}{2}\right)^2 - 2\cdot 3\cdot\frac{3}{2}\cdot\cos 120° = \frac{63}{4},\quad \overline{AB} = \frac{3\sqrt{7}}{2} \cong 3.97\ \text{cm}$$

**June 5-12:** The rhombicosidodecahedron is an Archimedean polyhedron that has 62 faces that are 30 squares, 12 regular pentagons, and 20 equilateral triangles. Determine the number of vertices



**Solution:** The rhombicosidodecahedron is a convex polyhedron therefore it fulfils Euler's formula: The number of faces plus the number of vertices is equal to the number of edges plus 2: C + V = A + 2.

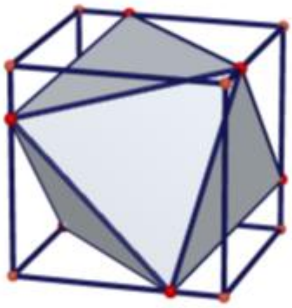
The edges are formed by the intersection of two sides of the polyhedral that make up the faces. So the number of edges is equal to half the number of sides that make up the polygons that make up the faces.

$$A = \frac{4\cdot 30 + 5\cdot 12 + 3\cdot 20}{2} = 120,\quad 62 + V = 120 + 2,\quad V = 60$$

**June 7-8:** A regular octahedron with vertices at six edges of the cube has been inscribed in a cube with edge a (see attached figure).

a) Calculate the edge of the octahedron

b) Calculate the proportion between the volumes of the octahedron and the cube



**Solution:** Let $\overline{AB} = a$, the edge of the cube. Be

$$\overline{AP} = \overline{AQ} = \overline{AR} = x$$

The edge of the regular octahedron is:

$$\overline{PQ} = x\sqrt{2}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{RA'S}$ :

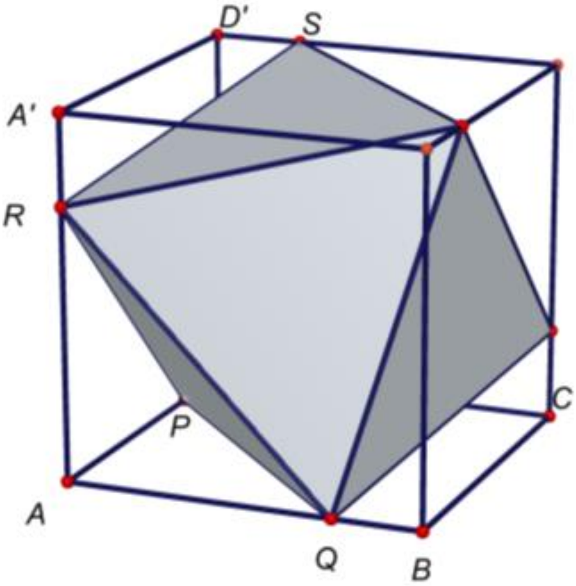
$$\overline{RS} = \sqrt{a^2 + 2(a-x)^2}$$

Matching the edges:

$$\sqrt{a^2 + 2(a-x)^2} = x\sqrt{2},\quad x = \frac{3}{4}a$$

The edge of the octahedron is:

$$\overline{PQ} = \frac{3\sqrt{2}}{4}a$$




The volume of the cube is:

$$V_{cube} = a^3$$

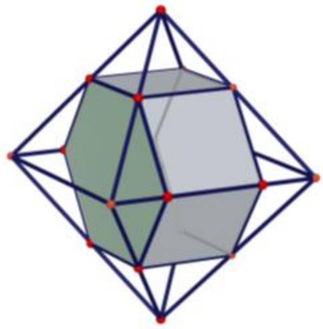
The volume of the octahedron is:

$$V_{\text{octahedron}} = \frac{\sqrt{2}}{3}\overline{PQ}^3 = \frac{\sqrt{2}}{3}\left(\frac{3\sqrt{2}}{4}a\right)^3 = \frac{9}{16}a^3$$

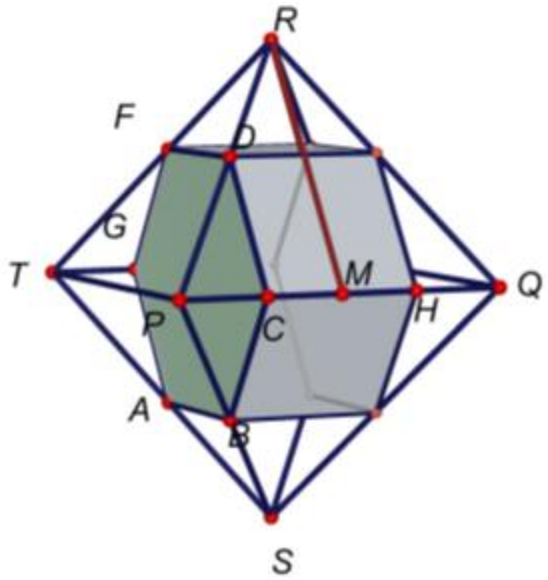
The ratio between the volumes is:

$$\frac{V_{\text{octahedron}}}{V_{\text{cube}}} = \frac{9}{16}$$

**June 9-16:** In a regular octahedron, a right hexagonal prism has been inscribed with all its edges the same. Determine the ratio between the volumes of the prism and the octahedron. (The hexagonal prism is not regular)



**Solution:** Be the regular octahedron of edge $\overline{PQ} = a$. Be ABCDFG the base of the prism. Let $\overline{CD} = \overline{CH} = x$, edges of the prism. (Note that the edges of the base are not equal). We will have:




$$\overline{PC} = \frac{a-x}{2}, \angle DPC = 60°, \text{tag}(60°) = \frac{x}{\frac{a-x}{2}} = \sqrt{3},$$

$$x = (2\sqrt{3}-3)a$$

Note that TSQR is a square with side a and, therefore:

$$\overline{RS} = a\sqrt{2}$$

The volume of the octahedron is:

$$V_{\text{octahedron}} = \frac{1}{3} \cdot a^2 \cdot a\sqrt{2} = \frac{\sqrt{2}}{3}a^3$$

We calculate the area of the base of the prism, we will have:

$$\overline{CG} = a, \overline{AB} = \overline{FC} = (2\sqrt{3}-3)a$$

(therefore, the base is not a regular hexagon). Since $\overset{\Delta}{PMR}$ is a triangle 30°-60°-90°:

$$\overline{MR} = \frac{\sqrt{3}}{2}a$$

On the other hand, the triangles $\overset{\Delta}{BCD}, \overset{\Delta}{SMR}$ are similar and when applying Thales, we have:

$$\frac{\overline{BD}}{\overline{RS}} = \frac{\overline{CD}}{\overline{MR}}, \qquad \frac{\overline{BD}}{a\sqrt{2}} = \frac{(2\sqrt{3}-3)a}{\frac{\sqrt{3}}{2}a}, \qquad \overline{BD} = (4\sqrt{2}-2\sqrt{6})a$$

Finally:

$$S_{ABCDFG} = 2 \cdot S_{CDFG} = \frac{\overline{CG} + \overline{DF}}{2}\overline{BD} = \frac{2\sqrt{3} - 2}{2}\left(4\sqrt{2} - 2\sqrt{6}\right)a^2 = \left(6\sqrt{6} - 10\sqrt{2}\right)a^2$$

And, the volume of the prism is:

$$V_{prism} = \left(6\sqrt{6} - 10\sqrt{2}\right)\left(2\sqrt{3} - 3\right)a^3 = \left(66\sqrt{2} - 38\sqrt{6}\right)a^3$$

The ratio between the volumes is:

$$\frac{V_{prism}}{V_{octahedron}} = \frac{66\sqrt{2} - 38\sqrt{6}}{\frac{\sqrt{2}}{3}} = 6\left(33 - 19\sqrt{3}\right)$$

**June 10-11:** Inside a cube a regular hexagonal dipyramid has been inscribed. Determine the ratio between the volumes of the dipyramid and the cube. Determine the ratio between the areas of the dipyramid and the cube

**Solution:** Let $\overline{AB} = a$ the edge of the cube. The volume and area of the cube are:

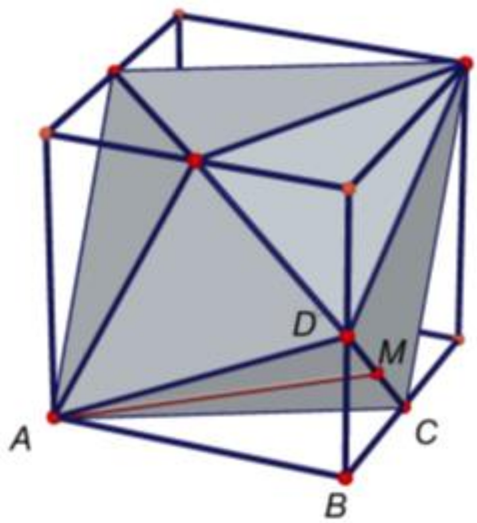
$$V_{cube} = a^3,\ \ S_{cube} = 6a^2$$

The volume of the dipyramid is equal to the volume of the cube minus six tetrahedral ABCD.

$$V_{dipyramid} = a^3 - 6 \cdot \frac{1}{3} \cdot \frac{1}{2}a \cdot \frac{1}{2}a \cdot \frac{1}{2}a = \frac{3}{4}a^3$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{BCD}$:

$$\overline{CD} = \frac{\sqrt{2}}{2}a$$




Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ABC}$

$$\overline{AC} = \frac{\sqrt{5}}{2}a$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ACM}$:

$$\overline{AM} = \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(\frac{\sqrt{2}}{4}\right)^2} = \frac{3\sqrt{2}}{4}a$$

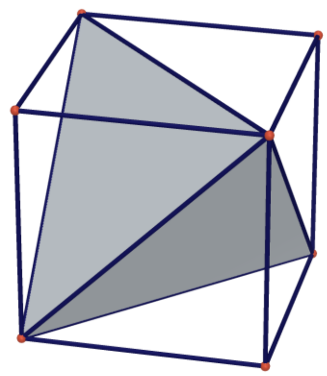
The area of the dipyramid is equal to twelve times the area of the triangle $\overset{\Delta}{ACD}$

$$S_{\text{dipyramid}} = 12 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} a \cdot \frac{3\sqrt{2}}{4} a = \frac{9}{2} a^2$$

Therefore:

$$\frac{V_{\text{dipyramid}}}{V_{\text{cube}}} = \frac{3}{4}, \quad \frac{S_{\text{dipyramid}}}{S_{\text{cube}}} = \frac{\frac{9}{2} a^2}{6a^2} = \frac{3}{4}$$

**June 14-15:** A tetrahedron is inscribed in a cube, as shown in the figure. Calculate the area of the tetrahedron and the ratio between the volume of the tetrahedron and the volume of the cube



**Solution 1:** Let a be the edge of the cube. Its volume will be:

$$V_{\text{cube}} = a^3$$

The volume of the tetrahedron is equal to the volume of the cube minus the volume of 4 tetrahedra that have 3 perpendicular cube edges

$$V_{\text{tetrahedron}} = a^3 - 4\left(\frac{1}{3} \cdot \frac{a^2}{2} \cdot a\right) = \frac{a^3}{3}$$

The volume ratio is:

$$\frac{V_{\text{tetrahedron}}}{V_{\text{cube}}} = \frac{1}{3}$$

**Solution 2:** Let a be the edge of the cube. Its volume will be:

$$V_{\text{cube}} = a^3$$

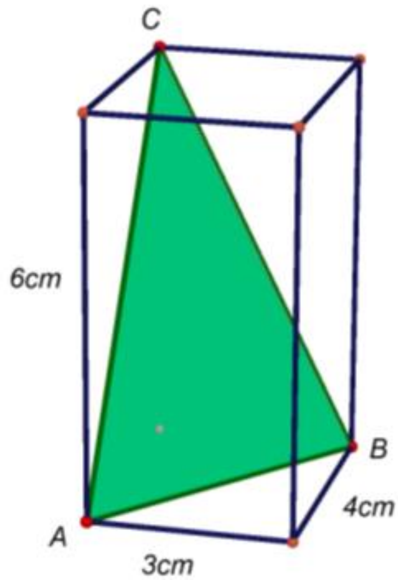
The tetrahedron is regular with edge $a\sqrt{2}$. And, hence, its volume is:

$$V_{\text{tetrahedron}} = \frac{x^3\sqrt{2}}{12} = \frac{\left(a\sqrt{2}\right)^3\sqrt{2}}{12} = \frac{a^3}{3}$$

The volume ratio is:

$$\frac{V_{\text{tetrahedron}}}{V_{\text{cube}}} = \frac{1}{3}$$

**June 17-24:** With the vertices of the orthohedron in the figure, the triangle ΔABC has been drawn. Calculate the measure of the sides of the triangle ΔABC. Calculate the angles of the triangle ΔABC. Calculate the area of the triangle ΔABC




**Solution:** Applying the Pythagorean theorem to the triangle $\overset{\Delta}{APC}$

$$\overline{AC} = 2\sqrt{13}$$

Applying the Pythagorean theorem to the triangle $\overset{\Delta}{AQB}$

$$\overline{AB} = 5$$

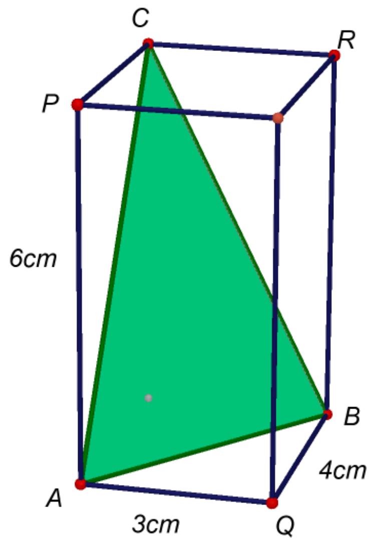
Applying the Pythagorean theorem to the triangle $\overset{\Delta}{BRC}$

$$\overline{BC} = 3\sqrt{5}$$

Applying the cosine theorem to the triangle $\overset{\Delta}{ABC}$

$$\left(3\sqrt{5}\right)^2 = 5^2 + \left(2\sqrt{13}\right)^2 - 2 \cdot 5 \cdot 2\sqrt{13} \cdot \cos A,$$

$$\cos A = \frac{8}{5\sqrt{13}}, A = \arccos\frac{8}{5\sqrt{13}} \approx 63°39'21"$$




$$5^2 = \left(3\sqrt{5}\right)^2 + \left(2\sqrt{13}\right)^2 - 2 \cdot 3\sqrt{5} \cdot 2\sqrt{13} \cdot \cos C, \quad \cos C = \frac{6}{\sqrt{65}}, \quad C = \arccos\frac{6}{\sqrt{65}} \approx 41°54'32"$$

$$B = 180° - (A + C) \approx 74°26'7"$$

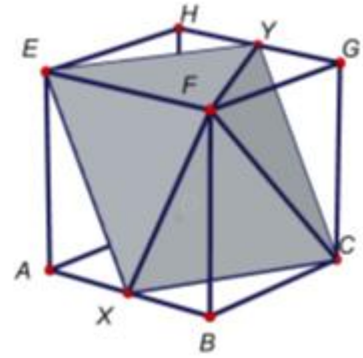
Finally, for the area:

$$S_{ABC} = \frac{1}{2}bc \cdot \sin A = \frac{1}{2}2\sqrt{13} \cdot 5 \cdot \sin 63°39'21" \approx 16.16 \text{ cm}^2$$

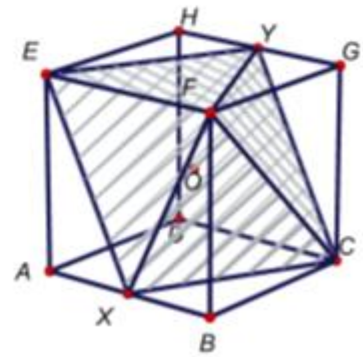
Or, using Heron's formula:

$$S_{ABC} = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$$

$$= \sqrt{\frac{2\sqrt{13}+5+3\sqrt{5}}{2} \cdot \frac{-2\sqrt{13}+5+3\sqrt{5}}{2} \cdot \frac{2\sqrt{13}-5+3\sqrt{5}}{2} \cdot \frac{2\sqrt{13}+5-3\sqrt{5}}{2}} = 3\sqrt{29}$$




**June 18-19:** Let ABCDEFGH be a cube of edge a. Let X and Y be the midpoints of edges AB and GH, respectively. The pyramid with base XCYE and vertex F is constructed. Calculate the measure of the segment XY, the area of the base XCYE and the volume of the pyramid XCYEF




**Solution:** Let O be the center of the cube. Point O belongs to the base XCYE of the pyramid. We will have:

$$\overline{XY} = \overline{BG} = \sqrt{a^2 + a^2} = a\sqrt{2}$$

after applying Pythagoras to the triangle ΔBCG.

The area of the base is twice the area of the triangle ΔCEX. When applying Pythagoras to the triangle ΔAEC:

$$\overline{CE} = \sqrt{\left(\sqrt{2}a\right)^2 + a^2} = a\sqrt{3}$$

$$\overline{OX} = \frac{1}{2} \cdot \overline{XY} = a\frac{\sqrt{2}}{2}$$

The area of the base XCYE is:

$$S_{XCYE} = 2\left(\frac{1}{2} \cdot \overline{CE} \cdot \overline{OX}\right) = \frac{\sqrt{6}}{2}a^2$$

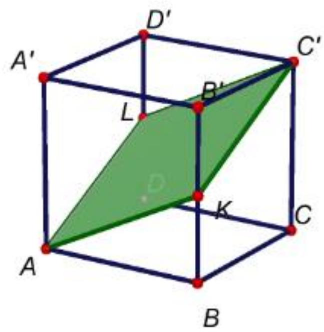
For the volume of the pyramid XCYEF, we will have:

$$X\left(\frac{a}{2}, 0, 0\right); \quad C(a, a, 0); \quad Y\left(\frac{a}{2}, a, a\right); \quad E(0, 0, a); \quad F(a, a, 0)$$

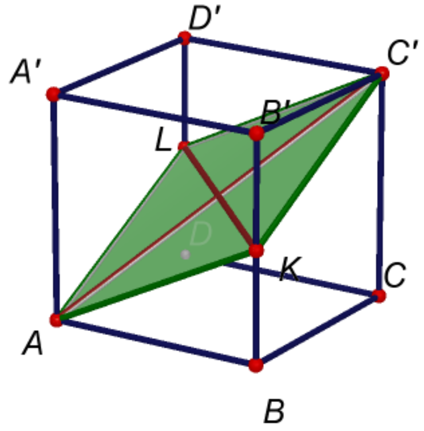
$$\text{Equation of the plane passing through X, C, Y y E: } \pi \equiv 2x - y + z = a$$

$$\text{pyramid height} = d(\pi, F) = \frac{|2 \cdot 0 - 0 + a - a|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2a}{\sqrt{6}}$$

$$V_{XCYEF} = \frac{1}{3} \cdot S_{XCYE} \cdot d(\pi, F) = \frac{1}{3} \cdot \frac{\sqrt{6}}{2}a^2 \cdot \frac{2a}{\sqrt{6}} = \frac{a^3}{3}$$




**June 21-28:** Let ABCDA'B'C'D 'be a cube with unit edge. Let K be the midpoint of edge BB '. Plane C'KA cuts edge DD 'at L. Find the angle formed by plane AKC' and face ABCD of the cube. Calculate the area of the AKC'L quadrilateral




**Solution:** Applying the Pythagorean theorem to the right and isosceles triangle ΔABC, we have:

$$\overline{AC} = \sqrt{2}$$

The angle formed by the plane AKC 'and the face ABCD of the cube is:

$$\alpha = \angle C'AC.$$

Applying trigonometric ratios to the right triangle $\overset{\Delta}{ACC'}$:

$$\alpha = \text{arctg}\frac{\overline{CC'}}{\overline{AC}} = \text{arctg}\frac{\sqrt{2}}{2} \approx 35°15'52"$$

Let's calculate the area of the quadrilateral AKC'L. We will have, L is the midpoint of the edge $\overline{DD'}$. AKC'L is a rhombus in which:

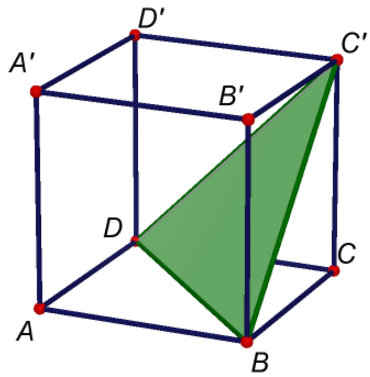
$$\overline{KL} = \overline{AC} = \sqrt{2}$$

And when applying the Pythagorean theorem to the right triangle ΔACC':

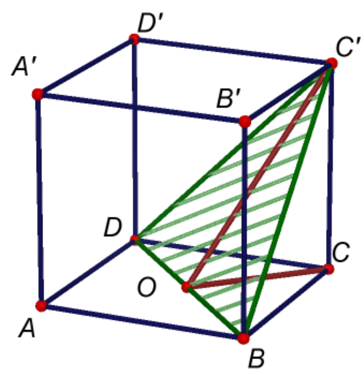
$$\overline{AC'} = \sqrt{\left(\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$$

The area of the AKC'L rhombus is:

$$S_{AKC'L} = \frac{1}{2}\overline{AC} \cdot \overline{KL} = \frac{\sqrt{6}}{2} \approx 1.22$$




**June 22-23:** Let ABCDA'B'C'D' be a cube with unit edge. Let us consider the plane that passes through BDC'. Find the angle formed by the plane that passes through BC'D and the face ABCD of the cube. Calculate area and perimeter of triangle ΔDBC'




**Solution:** Applying the Pythagorean theorem to the right and isosceles triangle $\overset{\Delta}{BCD}$:

$$\overline{AC} = \sqrt{2}$$

Let O be the center of square ABCD. The angle between the plane BC'D and the face ABCD of the cube is:

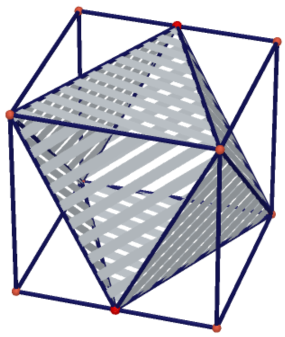
$$\alpha = \angle C'OC, \quad \overline{OC} = \frac{1}{2}\overline{BD} = \frac{\sqrt{2}}{2}$$

Applying trigonometric ratios to the right triangle $\overset{\Delta}{OCC'}$:

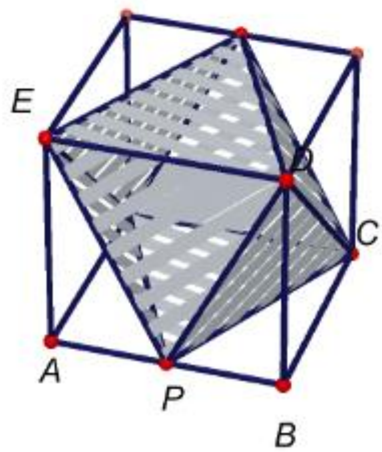
$$\alpha = \text{arctg}\frac{\overline{CC'}}{\overline{OC}} = \text{arctg}\sqrt{2} \approx 54°44'8"$$

Since the triangle $\overset{\Delta}{BC'D}$ is equilateral (three equal sides), its area is:

$$S_{BC'D} = \frac{\sqrt{3}}{4}\left(\sqrt{2}\right)^2 = \frac{\sqrt{3}}{2} \approx 0.87$$



**<u>June 25-26:</u>** An octahedron has been inscribed in a cube. Determine the ratio between their volumes and between their areas




**<u>Solution:</u>** Let $\overline{AB} = a$, the edge of the cube. The volume of the cube is $V_{cubo} = a^3$. The volume of the octahedron is equal to the volume of the cube minus four triangular pyramids of base the right triangle $\overset{\Delta}{PBC}$ and height $\overline{BD} = a$.

$$V_{octahedron} = a^3 - 4\left(\frac{1}{3}\frac{1}{2}\frac{1}{2}a \cdot a \cdot a\right) = \frac{2}{3}a^3$$

The ratio between the volumes is:

$$\frac{V_{octaedro}}{V_{cubo}} = \frac{2}{3}$$

The area of the cube is $S_{cube} = 6a^2$

The area of the octahedron is equal to four times the area of a triangle $\overset{\Delta}{PCD}$, with sides

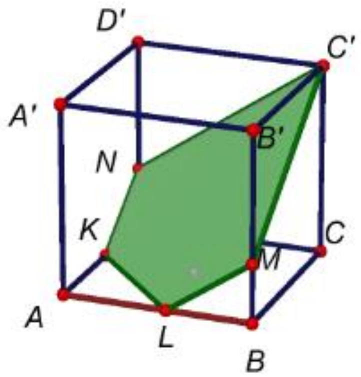
$$\overline{PC} = \overline{PD} = \frac{\sqrt{5}}{2}a, \overline{CD} = \sqrt{2}a$$

plus four times the area of a triangle $\overset{\Delta}{PDE}$. The height above the base $\overline{CD}$ of the triangle $\overset{\Delta}{PCD}$, measures $\frac{\sqrt{3}}{2}a$

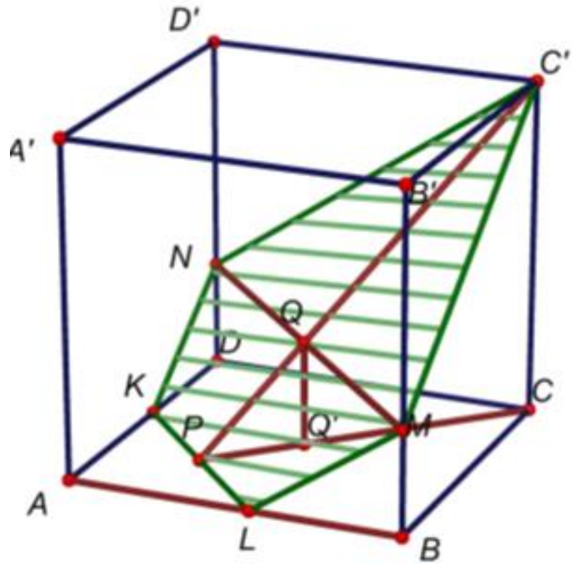
$$S_{octahedron} = 4\left(\frac{1}{2}a\sqrt{2}\frac{\sqrt{3}}{2}a\right) + 4\left(\frac{1}{2}a^2\right) = \left(2 + \sqrt{6}\right)a^2$$

The ratio between the volumes is:

$$\frac{S_{octahedron}}{S_{cube}} = \frac{2 + \sqrt{6}}{6}$$




**June 29-30:** Let ABCDA'B'C'D' be a cube with unit edge. Let K and L be the midpoints of edges AD and AB. Plane C'KL cuts edges BB' and DD' at M and N, respectively. Calculate the angle formed by the plane KLC' and the face ABCD of the cube. Calculate the area of the KLMC'N pentagon




**Solution:** Let P and Q be the midpoints of the segments $\overline{\mathrm{KL}}, \overline{\mathrm{NM}}$, respectively. The angle formed by the plane KLC' and the face ABCD of the cube is:

$$\alpha = \angle \mathrm{C'PC}$$

Applying the Pythagorean theorem to the right and isosceles triangle $\overset{\Delta}{\mathrm{APL}}$:

$$\overline{\mathrm{AP}} = \frac{\sqrt{2}}{4}, \qquad \overline{\mathrm{KL}} = \frac{\sqrt{2}}{2}.$$

Applying the Pythagorean theorem to the right and isosceles triangle $\overset{\Delta}{\mathrm{ABC}}$

$$\overline{\mathrm{AL}} = \frac{1}{2}, \qquad \overline{\mathrm{AC}} = \sqrt{2}, \qquad \overline{\mathrm{PC}} = \frac{3\sqrt{2}}{4}$$

Applying trigonometric ratios to the right triangle $\overset{\Delta}{\mathrm{PCC'}}$

$$\alpha = \operatorname{arctg}\frac{\overline{\mathrm{CC'}}}{\overline{\mathrm{PC}}} = \operatorname{arctg}\frac{2\sqrt{2}}{3} \approx 43°18'50''$$

Moreover:

$$\overline{\mathrm{MN}} = \sqrt{2}.$$

The projection Q' of point Q on face ABCD is the center of this face:

$$\overline{\mathrm{PQ'}} = \overline{\mathrm{AP}} = \frac{\sqrt{2}}{4}$$

Applying Thales' Theorem to Similar Triangles $\overset{\Delta}{\mathrm{PCC'}}, \overset{\Delta}{\mathrm{PQ'Q}}$:

$$\frac{\overline{\mathrm{QQ'}}}{1} = \frac{\frac{\sqrt{2}}{4}}{\frac{3\sqrt{2}}{4}}, \quad \overline{\mathrm{MB}} = \overline{\mathrm{QQ'}} = \frac{1}{3}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{PCC'}}$:

$$\overline{\mathrm{PC'}} = \sqrt{1^2 + \left(\frac{3\sqrt{2}}{4}\right)^2} = \frac{\sqrt{34}}{4}$$

Applying Thales' Theorem to Similar Triangles $\overset{\Delta}{\mathrm{PCC'}}, \overset{\Delta}{\mathrm{PQ'Q}}$:

$$\frac{\overline{\mathrm{PQ}}}{\frac{\sqrt{34}}{4}} = \frac{1}{3}, \qquad \overline{\mathrm{PQ}} = \frac{\sqrt{34}}{12}, \qquad \overline{\mathrm{QC'}} = \frac{\sqrt{34}}{6}$$

The area of the pentagon KLMC'N is equal to the sum of the areas of the trapezoid KLMN and the triangle $\overset{\Delta}{\mathrm{NMC'}}$

$$\mathrm{S}_{\mathrm{KLMC'N}} = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + \sqrt{2}\right)\frac{\sqrt{34}}{12} + \frac{1}{2}\sqrt{2}\frac{\sqrt{34}}{6} = \frac{7\sqrt{17}}{24} \approx 1.20$$