SOLUTIONS JULY 2021

HOLIDAYS. PROBLEMS TO NOT LOSE THE "TOUCH". AUTHORS: COLLECTIVE "CONCURSO DE PRIMAVERA". RICARD PEIRÓ I ESTRUCH

https://www.concursoprimavera.es/#concurso

July 1: As always the same (different) letters correspond to the same (different) digits

$$\begin{array}{r} N\ O\ T\ A\ R \\ \times\ 4 \\ \hline R\ A\ T\ O\ N \end{array}$$

Solution: Let's look at the last multiplication (Fig. 1). We will have:

4·N + (we carry) = R $\Rightarrow$ N = 1 o 2.

But N = 1 is impossible since no multiple of 4 ends in 1 (Fig. 2).

Fig. 1

$$\begin{array}{r} N\ O\ T\ A\ R \\ \times\ 4 \\ \hline R\ A\ T\ O\ N \end{array}$$

Fig. 2

$$\begin{array}{r} 1\ O\ T\ A\ R \\ \times\ 4 \\ \hline R\ A\ T\ O\ 1 \end{array}$$

Therefore, it must be N = 2. Also in this last multiplication we do not carry any of the previous multiplication (Fig. 3) because if we carry one (Fig. 4) we have that 9 · 4 does not end in 2

Fig. 3

$$\begin{array}{r} 2\ O\ T\ A\ 8 \\ \times\ 4 \\ \hline 8\ A\ T\ O\ 2 \end{array}$$

Fig. 4

$$\begin{array}{r} 2\ O\ T\ A\ 9 \\ \times\ 4 \\ \hline 9\ A\ T\ O\ 2 \end{array}$$

Now let's look at the penultimate multiplication (Fig. 5)

4·O + (we carry) = A $\Rightarrow$ O = 1 o 2. But O $\neq$ 2 = N $\Rightarrow$ O = 1

Fig. 5

$$\begin{array}{r} 2\ O\ T\ A\ 8 \\ \times\ 4 \\ \hline 8\ A\ T\ O\ 2 \end{array}$$

Fig. 6

$$\begin{array}{r} 2\ 1\ T\ A\ 8 \\ \times\ 4 \\ \hline 8\ A\ T\ 1\ 2 \end{array}$$

Now let's look at the second multiplication (Fig. 7). We will have (we carry 3 of the first multiplication):

4·A + 3 = 10x + 1 $\Rightarrow$ 4·A = 10x -2 $\Rightarrow$ 4·A ends at 8.

Hence, that, A = 2 or 7. But A $\neq$ 2 = N $\Rightarrow$ A = 7 (Fig. 8)

Fig. 7

$$\begin{array}{r} 2\ 1\ T\ A\ 8 \\ \times\ 4 \\ \hline 8\ A\ T\ 1\ 2 \end{array}$$

Fig. 8

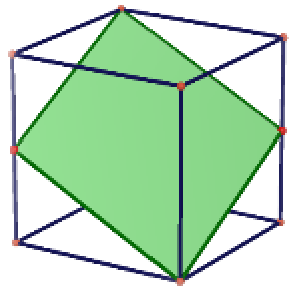
$$\begin{array}{r} 2\ 1\ T\ 7\ 8 \\ \times\ 4 \\ \hline 8\ 7\ T\ 1\ 2 \end{array}$$

Finally, (we carry 3 from the second multiplication and we need to carry 3 for the fourth multiplication):

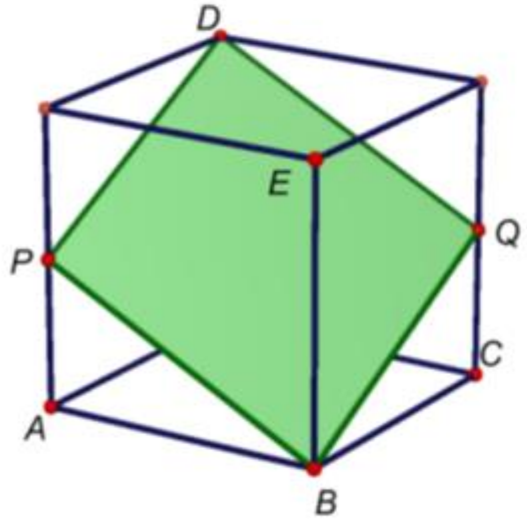
4·T + 3 = 30 + T $\Rightarrow$ 3T = 27 $\Rightarrow$ T = 9 (Fig. 9)

Fig. 9

$$\begin{array}{r} 2\ 1\ 9\ 7\ 8 \\ \times\ 4 \\ \hline 8\ 7\ 9\ 1\ 2 \end{array}$$



July 2-3: The cube in the figure has unity edge. The shaded quadrilateral has two opposite vertices at vertices of the cube and the other two vertices at midpoints of edges of the cube. Classify the quadrilateral, find its angles and sides, and its area



Solution: Let $\overline{\mathrm{AB}} = 1$ the edge of the cube. $\overline{\mathrm{AP}} = \frac{1}{2}$.
Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{PAB}}$:

$$\overline{\mathrm{PB}} = \overline{\mathrm{BQ}} = \overline{\mathrm{DQ}} = \overline{\mathrm{DP}} = \frac{\sqrt{5}}{2}.$$

Therefore, the quadrilateral PBQD is a rhombus.
Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{ABE}}$:

$$\overline{\mathrm{AE}} = \overline{\mathrm{AC}} = \overline{\mathrm{PQ}} = \sqrt{2}.$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{\mathrm{BED}}$:

$$\overline{\mathrm{BD}} = \sqrt{3}.$$

$\overline{\mathrm{BD}} \neq \overline{\mathrm{PQ}}$, and therefore PBQD is not a square.

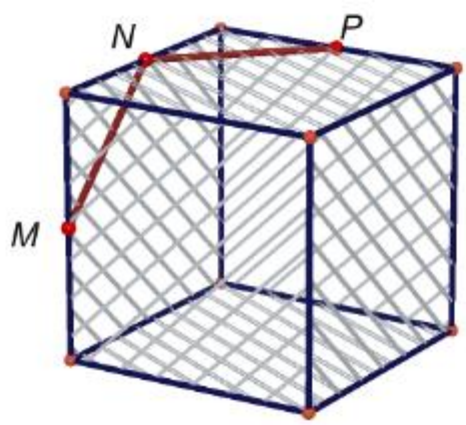
Let $\alpha = \angle\mathrm{BPD} = \angle\mathrm{BQD}$. Applying the cosine theorem to the triangle $\overset{\Delta}{\mathrm{PBD}}$:

$$\left(\sqrt{3}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2 - 2\cdot\left(\frac{\sqrt{5}}{2}\right)\left(\frac{\sqrt{5}}{2}\right)\cos\alpha, \quad \cos\alpha = \frac{-1}{5}, \quad \alpha = \arccos\frac{-1}{5} = 101°32'13"$$

$$\angle\mathrm{PBQ} = \angle\mathrm{PDQ} = 180° - \alpha = 78°27'47"$$

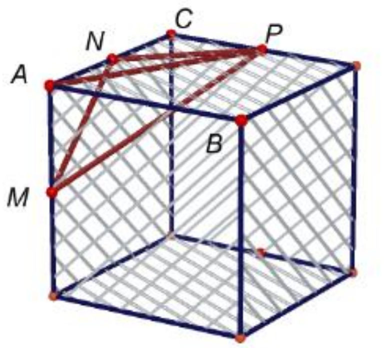
The area of the rhombus PBQD is:

$$S_{\mathrm{PBQD}} = \frac{1}{2}\overline{\mathrm{PQ}}\cdot\overline{\mathrm{BD}} = \frac{1}{2}\sqrt{2}\cdot\sqrt{3} = \frac{\sqrt{6}}{2}$$



July 5: Let M, N and P be the midpoints of three consecutive edges of a cube. Find the angle $\angle\mathrm{MNP}$

Solution: Let $\overline{AB} = a$, the edge of the cube. $\alpha = \angle MNP$. Obviously:



$$\overline{MN} = \overline{NP} = \frac{\sqrt{2}}{2}a$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ACP}$.

$$\overline{AP} = \frac{\sqrt{5}}{2}a$$

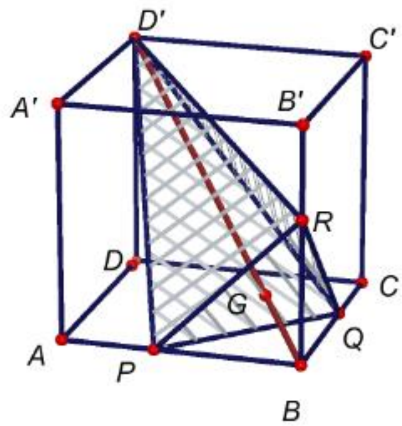
Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{MAP}$.

$$\overline{MP} = \frac{\sqrt{6}}{2}a$$

Applying the cosine theorem to the triangle$\overset{\Delta}{MNP}$

$$\left(\frac{\sqrt{6}}{2}a\right)^2 = \left(\frac{\sqrt{2}}{2}a\right)^2 + \left(\frac{\sqrt{2}}{2}a\right)^2 - 2\left(\frac{\sqrt{2}}{2}a\right)^2 \cos\alpha, \quad \cos\alpha = -\frac{1}{2}, \quad \alpha = \angle MNP = 120°$$

July 6-7: Let ABCDA'B'C'D' be a cube of edge a. Let P, Q and R be points on the edges AB, BC and BB', respectively, such that BP = BQ = BR = x. Determine the volume of the PQRD' tetrahedron as a function of x and a.



Solution: Applying the Pythagorean theorem to triangles ΔADB, ΔDBD', ΔPBQ, ΔPBR and ΔRBQ, we arrived to:

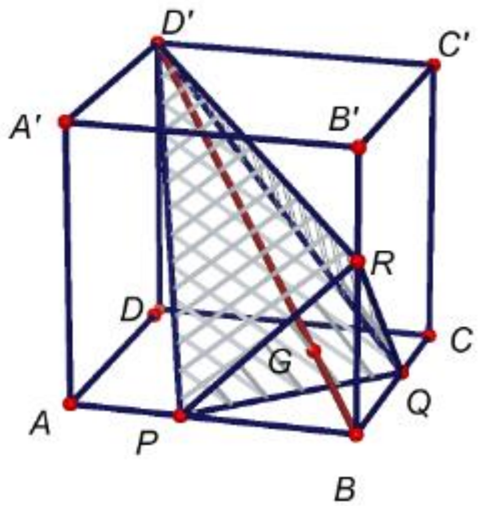
$$\overline{PQ} = \overline{QR} = \overline{PR} = x\sqrt{2}, \qquad \overline{DB} = a\sqrt{2}, \qquad \overline{BD'} = a\sqrt{3}$$

Let G be the center of gravity of the equilateral triangle ΔPQR. Let $h = \overline{GB}$ the height of the tetrahedron PBQR on the base ΔPQR. The volume of the PBQR tetrahedron is:

$$V_{PBQR} = \frac{1}{3}\cdot\frac{1}{2}\cdot x^3 = \frac{1}{3}\cdot\left(\frac{\sqrt{3}}{4}\left(x\sqrt{2}\right)^2\right)\cdot h$$

Solving for h, we arrive at:

$$h = \frac{\sqrt{3}}{3}x$$



Therefore, the height of the PQRD' tetrahedron above the base ΔPQR is:

$$\overline{GD'} = a\sqrt{3} - \frac{\sqrt{3}}{3}x$$

And, the volume of the PQRD' tetrahedron is:

$$V_{PQRD'} = \frac{1}{3}\cdot\frac{\sqrt{3}}{4}\cdot\left(x\sqrt{2}\right)^2\cdot\left(a\sqrt{3}-\frac{\sqrt{3}}{3}x\right) = \frac{1}{2}\cdot x^2\cdot\left(a-\frac{1}{3}x\right) = \frac{1}{6}\cdot(3a-x)\cdot x^2$$

July 8-15: Carmen is retiring this year and her students have decided to give her each a pentagon or a hexagon. Carmen has counted 282 edges and 49 polygons. If those who gave her hexagons had given her pentagons and those who gave her pentagons had given her hexagons, how many edges would she have?

Solution: Let x (y) be the number of pentagons (hexagons). From the information in the statement, we have:

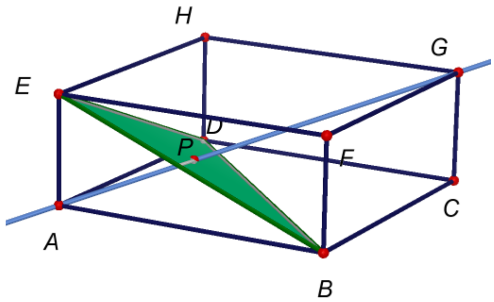
$$\left.\begin{array}{l} 5x+6y=282 \\ x+y=49 \end{array}\right\}$$

From the second equation we solve for y and substitute in the first one with what:

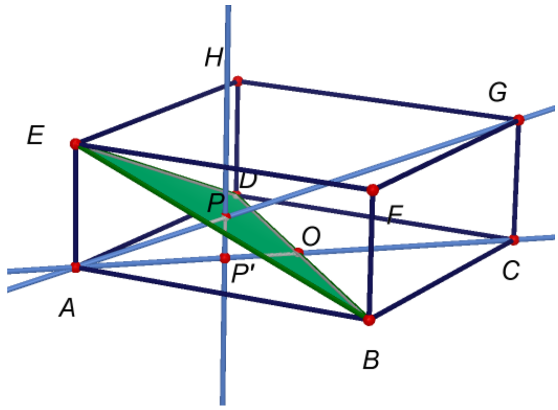
$$5x+6(49-x)=282,\quad x=12,\quad y=49-12=37$$

The answer to the question posed is: 6·12 + 5·37 = 257

July 9-10: Given the orthohedron ABCDEFGH, the diagonal AG intersects the triangle ΔBDE at the point P. Show that P is the center of gravity of the triangle ΔBDE



Solution 1: Let $\overline{AB}=a, \overline{BC}=b, \overline{AE}=c$ the edges of the orthohedron. Let O be the center of face ABCD. O is the midpoint of the diagonals of rectangle ABCD. Let P' be the projection of P on the face ABCD. Points E, A, P, P', O, C, G are coplanar. Points E, P, O are aligned.



Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ABC}$

$$\overline{AC}=\sqrt{a^2+b^2}$$

Let $x=\overline{AP'}$

$$\overline{OP'}=\frac{1}{2}\sqrt{a^2+b^2}-x$$

The right triangles $\overset{\Delta}{APP'}$, $\overset{\Delta}{ACG}$ are similar. Applying Thales' theorem:

$$\frac{c}{\sqrt{a^2+b^2}}=\frac{\overline{PP'}}{x}\quad (1)$$

The right triangles $\overset{\Delta}{EAO}$, $\overset{\Delta}{PP'O}$ are similar. Applying Thales' theorem:

$$\frac{2c}{\sqrt{a^2+b^2}}=\frac{\overline{PP'}}{\frac{1}{2}\sqrt{a^2+b^2}-x}\quad (2)$$

Dividing the expressions ((1): (2)):

$$\frac{1}{2}=\frac{\frac{1}{2}\sqrt{a^2+b^2}-x}{x}$$

Solving the equation:

$$x=\frac{1}{3}\sqrt{a^2+b^2}$$

Applying Thales' theorem to right triangles $\overset{\Delta}{EAO}$, $\overset{\Delta}{PP'O}$

$$\frac{\overline{EP}}{\overline{OP}}=\frac{\overline{AP'}}{\overline{OP'}}=\frac{\frac{\sqrt{a^2+b^2}}{3}}{\frac{\sqrt{a^2+b^2}}{2}-\frac{\sqrt{a^2+b^2}}{3}}=\frac{\frac{\sqrt{a^2+b^2}}{3}}{\frac{\sqrt{a^2+b^2}}{6}}=2$$

Therefore, P is the center of gravity of the triangle $\overset{\Delta}{BDE}$

<u>Solution 2 (Nick Kalapodis @NickKalapodis):</u> Let A(0, 0, 0); B(a, 0, 0); C(a, b, 0); D(0, b, 0), E(0,0,c); F(a, 0, c); G(a, b, c) and H(0, b, c). The equation of the plane that passes through E (0,0, c); B (a, 0, 0) and D (0, b, 0) is:

$$\begin{vmatrix} x-0 & y-0 & z-c \\ a-0 & 0-0 & 0-c \\ 0-0 & b-0 & 0-c \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix}=0 \Rightarrow x\begin{vmatrix} 0 & -c \\ b & -c \end{vmatrix}-a\begin{vmatrix} y & z-c \\ b & -c \end{vmatrix}=0$$

$$\Rightarrow \ bcx+cay+abz=abc \ \Rightarrow \ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \ (1)$$

The equation of the line through A (0, 0, 0) and G (a, b, c) is:

$$\frac{x-0}{a-0}=\frac{y-0}{b-0}=\frac{z-0}{c-0} \ \Rightarrow \ \frac{x}{a}=\frac{y}{b}=\frac{z}{c} \ (2)$$

The coordinates of the point that belongs to the plane (1) and the line (2) is

$$P\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

The center of gravity of the triangle ΔBDE is:

$$G\left(\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}\right)=\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)=P$$

<u>Julio 12-13:</u> The tutors of 1D and 1E have received stickers to distribute among their students. Each student received 25 stickers and there were 8 left over. As 4 students did not want to take stickers, they distributed the remaining stickers and touched 3 more stickers than they had. How many students took trading cards?

<u>Solution:</u> As four students did not want stickers, the remaining stickers were: 4 · 25 + 8 = 108. These were distributed among the x students of the two groups who did want stickers by touching each of them 3 more stickers. Later:

$$x=\frac{108}{3}=36$$

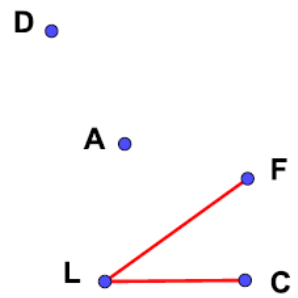
July 14: To fill the cup to half its height, we have taken 1 second, how long will it take to fill the entire cup of the illustration?

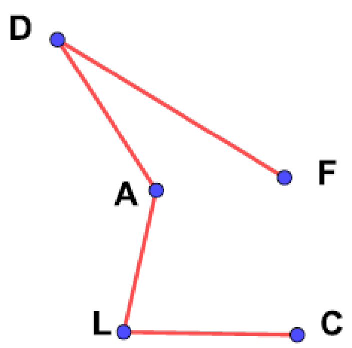
Solution: The triangles generated by the section of the cone generated by a plane that passes through a diameter of the base of the cone and through the vertex of the cone are similar, with a ratio of proportionality 1: 2. Therefore, the volume of the large cone is (23) eight times the volume of the small cone.

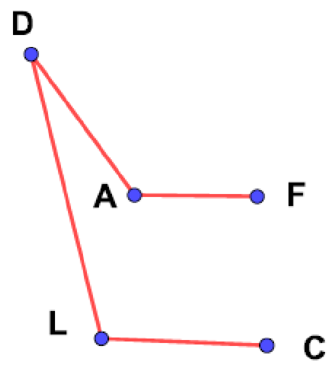
Therefore, it will take seven more seconds to fill the entire cup

July 16: In a committee of 5 people, Laia, Dani and Aitana know two people, while Carles and Ferran know only one. If Laia and Carles know each other, is it possible that Laia and Ferran know each other?

Solution: If we assume that Laia and Ferran are known, we will have the following attached graph that should be completed with two lines reaching A and two lines reaching D (note that in L, F and C there are already all the lines required in the statement) and this is impossible. The only possible graphs fulfilling the statement are the ones below and in them L and F are not connected







July 17: If we increase the sides of a square by a certain percentage, its area increases by 96%. By what percentage would its area have decreased if instead of lengthening the sides, I shortened them by that percentage?

Solution: Let x be the side of the square. If we increase the sides by r%, its length goes from x to $\left(1+\frac{r}{100}\right)x$ and its area goes from x^2 to $\left(1+\frac{r}{100}\right)^2 x^2$. From the statement we have:

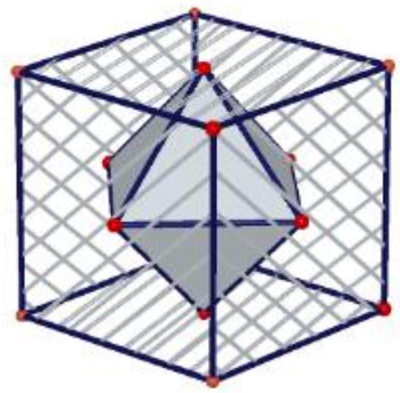
$$\left(1+\frac{r}{100}\right)^2 = 1{,}96; \quad 1+\frac{r}{100} = \sqrt{1{,}96}; \quad r = 40$$

If we decrease the sides of the square by 40%, its length goes from x to $\left(1-\frac{40}{100}\right)x = 0{,}6 \cdot x$ and its area changes from x^2 to $0.6^2 \cdot x^2$. From here:

$$0{,}6^2 = \left(1 - \frac{s}{100}\right); \quad 0{,}36 = 1 - \frac{s}{100}; \quad s = 64$$

That is, its area decreases by 64%

July 19-20: The octahedron that has the vertices at the centers of the faces of a cube is called the dual polyhedron of the cube. Calculate the ratio between the volumes of the cube and its dual octahedron



Solution: Let c be the edge of the cube. The diagonal of the regular octahedron is equal to the edge of the cube. The edge of the octahedron is

$$\frac{\sqrt{2}}{2}c$$

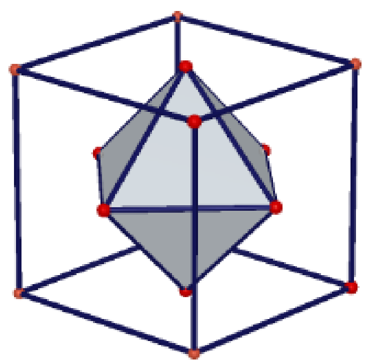
The volume of the octahedron is equal to twice the volume of a regular quadrangular pyramid with an edge of the base $\frac{\sqrt{2}}{2}c$ and height $\frac{1}{2}c$:

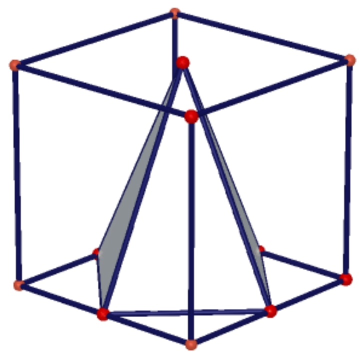
$$V_{octahedron} = 2 \cdot \left(\frac{1}{3} \cdot \left(\frac{\sqrt{2}}{2}c\right)^2 \frac{1}{2}c\right) = \frac{1}{6}c^3$$

The ratio between the volumes of the octahedron and the cube is:

$$\frac{V_{octahedron}}{V_{cube}} = \frac{\frac{1}{6}c^3}{c^3} = \frac{1}{6}$$

Note that the volume of the octahedron is equal to the volume of the pyramid based on a square that has the vertices at the midpoints of a face and a height equal to the edge of the cube.



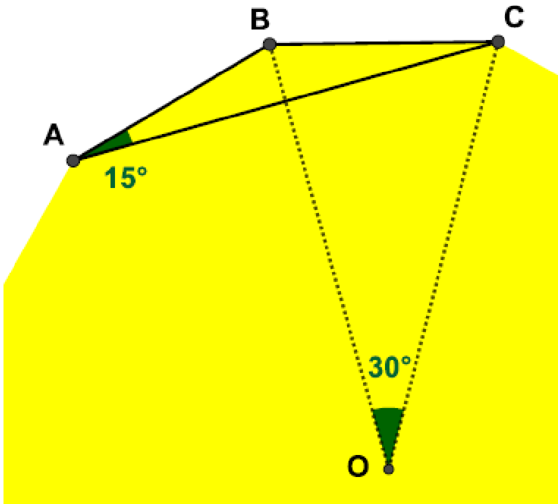


July 21: It has a regular polygon. If AB and BC are two consecutive edges and ∠BAC = 15°, how many sides does the polygon have?

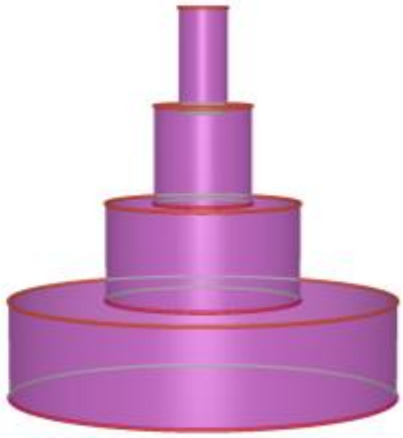
Solution: Let n be the number of sides of the regular polygon. From the statement ∠BAC = 15°. Let O be the center of the regular polygon (the center of the circumscribed circle). For the relationship between the angle inscribed in a circle and the central angle we will have, ∠BOC = 30°. But:

$$30° = \frac{360°}{n} \Rightarrow n = \frac{360°}{30°} = 12$$

That is, it is a dodecagon



July 22-29: In the figure, the lower cylinder has radius 1 and height 1. The upper cylinders have half the radius of the lower cylinder and height 1. Determine the volume of the 4 cylinders. Calculate the volume if there were 10 cylinders. Calculate the volume in the case of infinite cylinders



Solution: For the volume of the four cylinders:

$$V_4 = \pi \cdot 1^2 \cdot 1 + \pi\left(\frac{1}{2}\right)^2 \cdot 1 + \pi\left(\frac{1}{4}\right)^2 \cdot 1 + \pi\left(\frac{1}{8}\right)^2 \cdot 1 = \pi\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}\right) = \frac{85}{64}\pi \approx 4.172427743$$

The volumes of the cylinders form a geometric progression of the first term π and ratio $\frac{1}{4}$. The sum of the first 10 terms is:

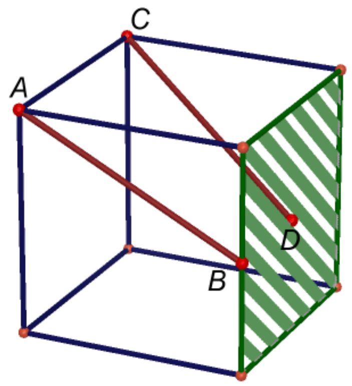
$$V_{10} = \frac{\pi\left(1 - \left(\frac{1}{4}\right)^{10}\right)}{1 - \frac{1}{4}} = \frac{349525}{262144}\pi \approx 4{,}18878621$$

The sum of the infinite terms of a geometric progression of first term π and ratio $\frac{1}{4}$ is:

$$V_{inf} = \frac{\pi}{1 - \frac{1}{4}} = \frac{4}{3}\pi \approx 4.188790205\ .$$

Note: The sum of the infinite volumes of the cylinders is equal to the volume of a sphere of radius 1.

July 23-24: In the figure, A and C are vertices of a cube of edge 1, B is the midpoint of the edge, and D is the center of the shaded face. Do the lines through A and B and the one through C and D intersect? If yes, at what point? Calculate the area of quadrilateral ABCD.

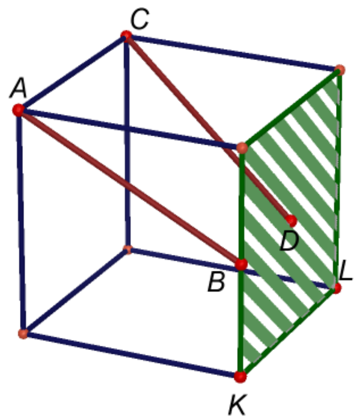


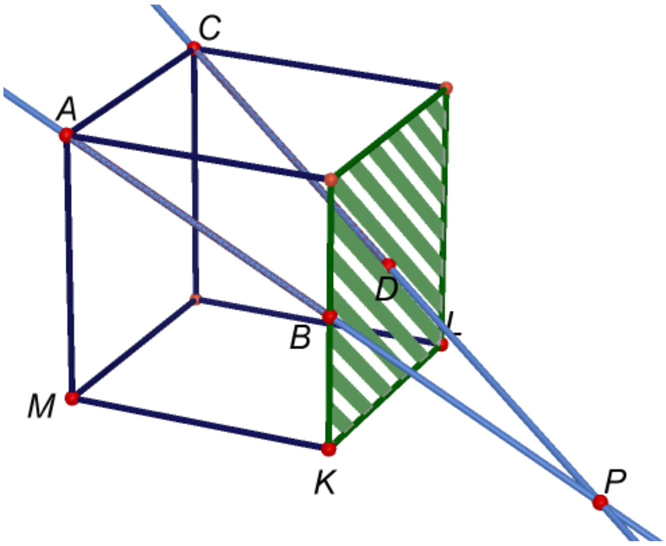
Solution: Segments $\overline{AC}$ and $\overline{BD}$ are parallel since edges $\overline{AC}$ and $\overline{KL}$ are parallel and $\overline{KL}$ and $\overline{BD}$ are parallel. Then, A, B, C, and D are parallel. So A, B, C, and D are coplanar. Therefore, lines AB and CD are secant. Obviously:

$$\overline{BD} = \frac{1}{2}\overline{AC}$$

Let P be the intersection of lines AB and CD. $\overline{BD}$ is the median parallel of triangle $\overset{\Delta}{ACP}$. P belongs to the plane formed by the points ABK.

$$\overline{AB} = \overline{BP}$$





Then, P belongs to the line MK and $\overline{MP} = 2 \cdot \overline{MK}$

ABDC is a right trapezoid $\angle CAB = 90°$.

$$\overline{AB} = \frac{\sqrt{5}}{2}$$

The area of the trapezoid ABDC is:

$$S_{ABDC} = \frac{1}{2}\left(1 + \frac{1}{2}\right)\frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{8}$$

July 26: How many digits does the number N have?

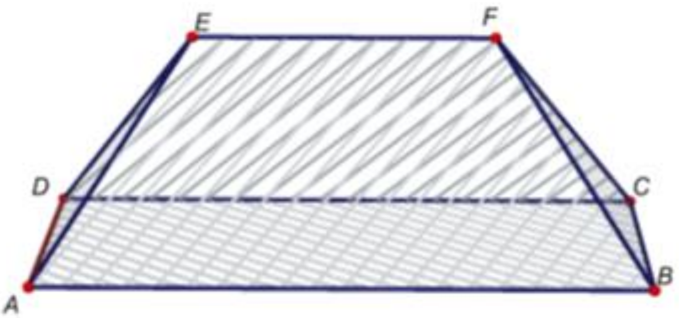
$$N = 16^{505} \cdot 3125^{404}$$

Solution: We will have:

$$N = (2^4)^{505} \cdot (5^5)^{404} = 2^{2020} \cdot 5^{2020} = (2 \cdot 5)^{2020} = 10^{2020}$$

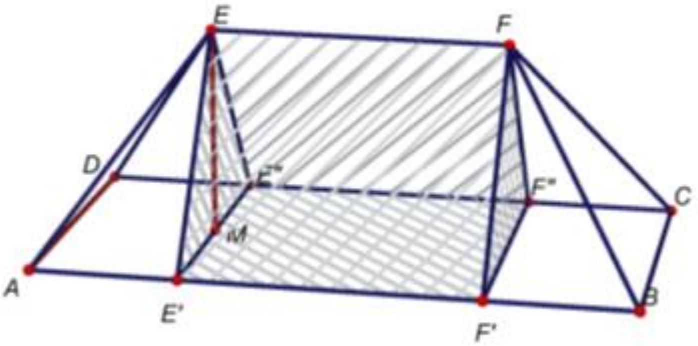
That is, N has 2021 digits: a 1 followed by 2020 zeros.

July 27-28: In the figure ABCD is a rectangle with AB = 20 and BC = 10. The triangles $\triangle ADE$ and $\triangle CBF$ are equal and equilateral. If EF = 10 and EF // AB, calculate body volume



Solution: Let E' be the projection of E on $\overline{AB}$. Let F' be the projection of F on $\overline{AB}$. Let E" be the projection of E onto $\overline{CD}$. Let F" be the projection of F onto $\overline{CD}$. Obviously:

$$\overline{AE'} = \overline{DE''} = \overline{BF'} = \overline{CF''} = \frac{\overline{AB} - \overline{EF}}{2} = 5$$



Applying the Pythagorean theorem to the right triangle $\triangle AE'E$:

$$\overline{EE'} = \sqrt{10^2 - 5^2} = 5\sqrt{3}.$$

Let M be the midpoint of the segment $\overline{E'E''}$.

$$\overline{E'M} = \frac{1}{2}\overline{BC} = 5.$$

Applying the Pythagorean theorem to the right triangle $\triangle E'EM$:

$$\overline{EM} = \sqrt{\left(5\sqrt{3}\right)^2 - 5^2} = 5\sqrt{2}.$$

The volume of the figure is equal to the sum of the volume of the prism EE'E''F'F"F of triangular base plus twice the volume of the pyramid AE'E"DE of rectangular base.

The volume of the prism EE'E"FF'F" is:

$$V_{EE'E''FF'F''} = \frac{1}{2}\overline{E'E''}\cdot\overline{EM}\cdot\overline{EF} = \frac{1}{2}10\cdot 5\sqrt{2}\cdot 10 = 250\sqrt{2}.$$

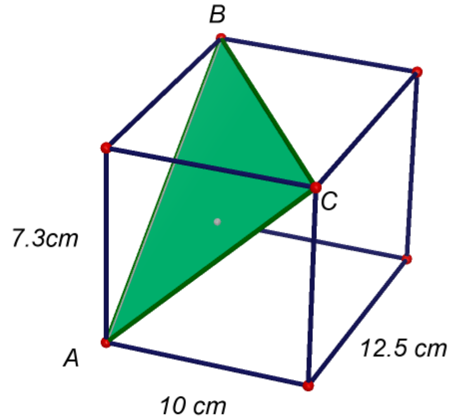
The volume of the pyramid AE'E"DE is:

$$V_{AE'E''DE} = \frac{1}{3}\overline{AE'}\cdot\overline{E'E''}\cdot\overline{EM} = \frac{1}{3}5\cdot 10\cdot 5\sqrt{2} = \frac{250}{3}\sqrt{2}.$$

The volume of the figure is:

$$V = 250\sqrt{2} + 2\left(\frac{250}{3}\sqrt{2}\right) = \frac{1250}{3}\sqrt{2}.$$

July 30-31: An orthohedron has edges of 12.5 cm, 10 cm, and 7.3 cm. Calculate the area of the triangle ΔABC



Solution: Applying the Pythagorean Theorem three times:

$\overline{AB}^2 = 12.5^2 + 7.3^2 = 209.54$ $\overline{AC}^2 = 10^2 + 7.3^2 = 153.29$ $\overline{BC}^2 = 12.5^2 + 10^2 = 256.25$

$\overline{AB} = \sqrt{12.5^2 + 7.3^2} \approx 14.48 \text{ cm}$ $\overline{AC} = \sqrt{10^2 + 7.3^2} \approx 12.38 \text{ cm}$ $\overline{BC} = \sqrt{12.5^2 + 10^2} \approx 16.01 \text{ cm}$

Applying the cosine theorem:

$A = \arccos\left(\frac{256.25 - 209.54 - 153.29}{-2\sqrt{209.54}\sqrt{153.29}}\right) \approx 72^o42'$ $B = \arccos\left(\frac{153.29 - 209.54 - 256.25}{-2\sqrt{209.54}\sqrt{256.25}}\right) \approx 47^o36'$

$C = 180^o - (A + B) \approx 59^o42'$.

The area of the triangle is:

$$S_{ABC} = \frac{\sqrt{209.54}\sqrt{153.29}\sin(72^o42')}{2} \approx 85.56 \text{ cm}^2 .$$