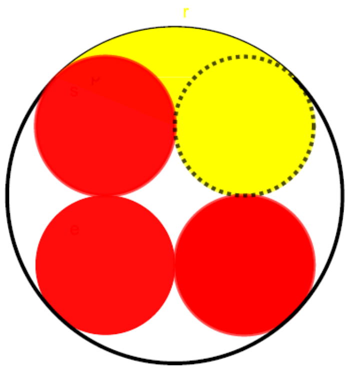
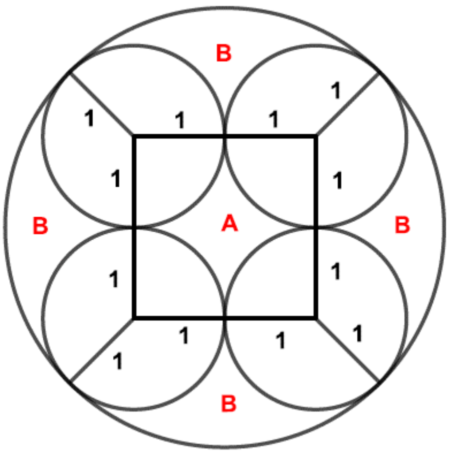
# SOLUTIONS NOVEMBER 2021

COLLECTION FOR THE PREPARATION OF THE OLYMPICS OF THE FESPM OF 1ESO AND 2ESO IN 2004 AND 2005. AUTHOR: JOSÉ COLÓN LACALLE. RETIRED TEACHER.



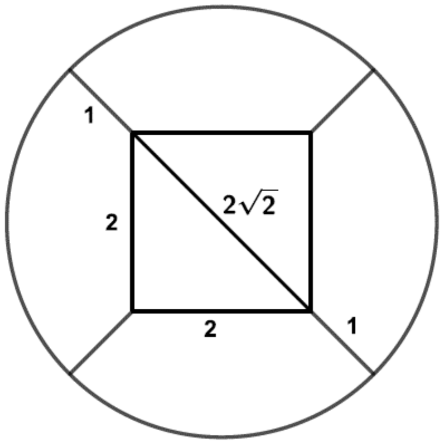
**November 1-2:** The figure shows four circles of radius 1 inscribed in a larger circle.

What is the area of the yellow zone?




**Solution:** We will designate by A and B the areas of the drawing on the left. By C and D, the circles of radius 1 and R and by E the square of side (1 + 1 =) 2.

The area of zone E ($A_E$) is the area of zone A ($A_A$) plus the area of four quadrants of radius 1 ($A_C$).




$$2R = 1 + 2\sqrt{2} + 1 \Rightarrow R = 1 + \sqrt{2}$$

The area of zone B ($A_B$) is the fourth part of the area of zone D ($A_D$) minus the area of zone A ($A_A$) minus the area of four zones C ($A_C$).

We will have:

$$A_A = A_E - A_C = 2 \cdot 2 - \pi \cdot 1^2 = 4 - \pi$$

$$A_B = \frac{A_D - A_A - 4 \cdot A_C}{4} = \frac{\pi\left(1+\sqrt{2}\right)^2 - 4 + \pi}{4} - \pi = \frac{\sqrt{2}}{2}\pi - 1$$

Therefore, the area of the yellow zone is the area of zone B ($A_B$) plus the area of a circle of radius 1, that is:

$$A_{\text{yellow}} = A_B + A_C = \frac{\sqrt{2}}{2}\pi - 1 + \pi = \left(1 + \frac{\sqrt{2}}{2}\right)\pi - 1$$

**November 3-10:** Three friends decide to sell orange granite at the entrance of a sports stadium. They bought oranges and sugar and paid €50. In addition, they paid €100 for the rental of tables, purchase of glasses and straws to sip. They calculated to get 250 glasses of granite. How much should they sell each cup for to make a 25% profit?

**Solution:** The money spent by the three friends is: (€100 + €50 =) €150. Earnings of 25% go up to:

$$\frac{25}{100} \cdot 150 = 37{,}50\ €$$

income should be

$$150\ € + 37{,}50\ € = 187{,}50\ €$$

The price of each glass should be:

$$\frac{187{,}50}{250} = 0{,}75\ €$$

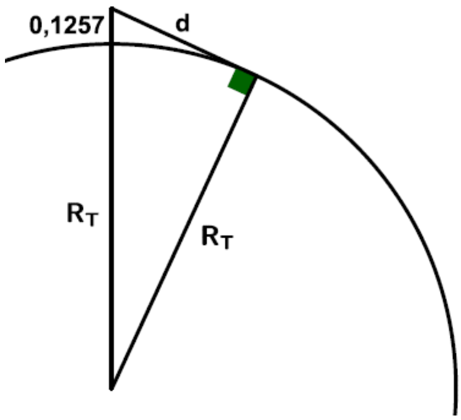
**November 4:** From the top of a lighthouse located 127.5 m above sea level, you can see the horizon. How far is it from the lighthouse, knowing that traveling around the world is 40,000 km?

**Solution:** From the statement we have:

$$2\pi R_T = 40000 \Rightarrow R_T = \frac{40000}{2\pi} = \frac{20000}{\pi}$$

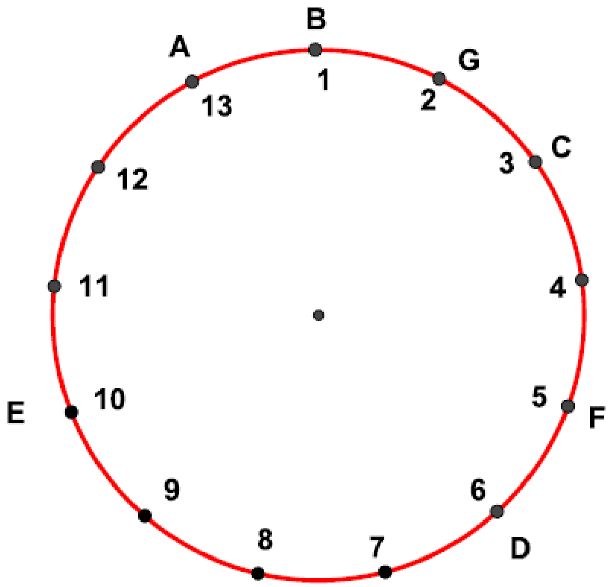
Applying Pythagoras:

$$d = \sqrt{(R_T + 0{,}1257)^2 - R_T^2} = \sqrt{2 \cdot R_T \cdot 0{,}1257 + 0{,}1257^2}$$
$$= \sqrt{2 \cdot \frac{20000}{\pi} \cdot 0{,}1257 + 0{,}1257^2}$$
$$\cong 40{,}0059\text{km}$$



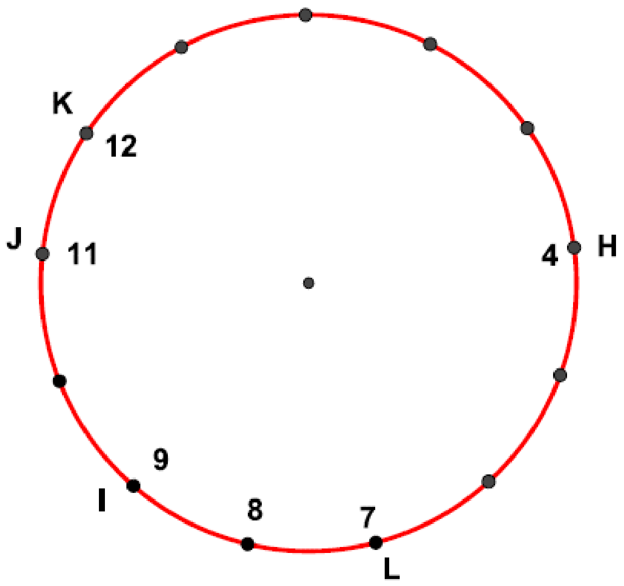

**November 5-6:** We have 12 half euro coins and one-euro coin. We put them in a circle. Starting with the coin you want, you have to count 13 coins to the right and the one in that place is eliminated. We count 13 coins again starting with the next one, to our right, from the one we just removed. We repeat this operation until only one coin is left. By what currency should we start counting so that the last one we withdraw is the one-euro coin?

**Solution:** This is a variant of the classic problem of Josephus.




We will play the game until there is only one-coin left. In that position we must leave the one-euro coin.

We start with coin 1. We remove coin 13 (A) (12 coins remain). We start counting with coin 1. We remove coin 1 (B) (there are 11 coins left). We start counting with coin 2. We remove coin 3 (C) (10 coins remain). We start counting from letter 4. We remove coin 6 (D) (9 coins remain). We start counting from coin 7. We remove coin 10 (E) (8 coins remain). We start counting from coin 11. We remove coin 5 (F) (7 coins remain). We start counting from coin 7. We remove coin 2 (G) (6 coins remain)




We start counting from coin 4. We remove coin 4 (H) (5 coins remain). We start counting from coin 7. We remove coin 9 (I) (4 coins remain). We start counting from coin 11. We remove coin 11 (J) (3 coins remain). We start counting with coin 12. We remove coin 12 (K) (2 coins remain): We start counting with coin 7. We remove coin 7 (L). That coin must be the one-euro coin. There is one coin left which is coin 8

**November 8:** What is the natural minor that divided by 2, 3, 4, 5 and 6 gives, respectively, the remainder 1, 2, 3, 4 and 5?

**Solution:** Consider N = lcm (2, 3, 4, 5, 6). Then N – 1 gives remainder 1 (2, 3, 4, 5) when divided by 2 (3, 4, 5, 6). Therefore, N – 1 is a number that meets the requirements of the statement. If there were another number p, less than N – 1 that fulfilled the above, we would have:

$$p < N - 1 \;\; \Rightarrow \;\; p + 1 < N$$

but, p + 1 would be a multiple of 2, 3, 4, 5 and 6; contradicting that N is the smallest of those multiples. We will then have that N – 1 is the number sought where N = lcm (2, 3, 4, 5, 6) = $2^2 \cdot 3 \cdot 5$ = 60. Then the number sought is 59.

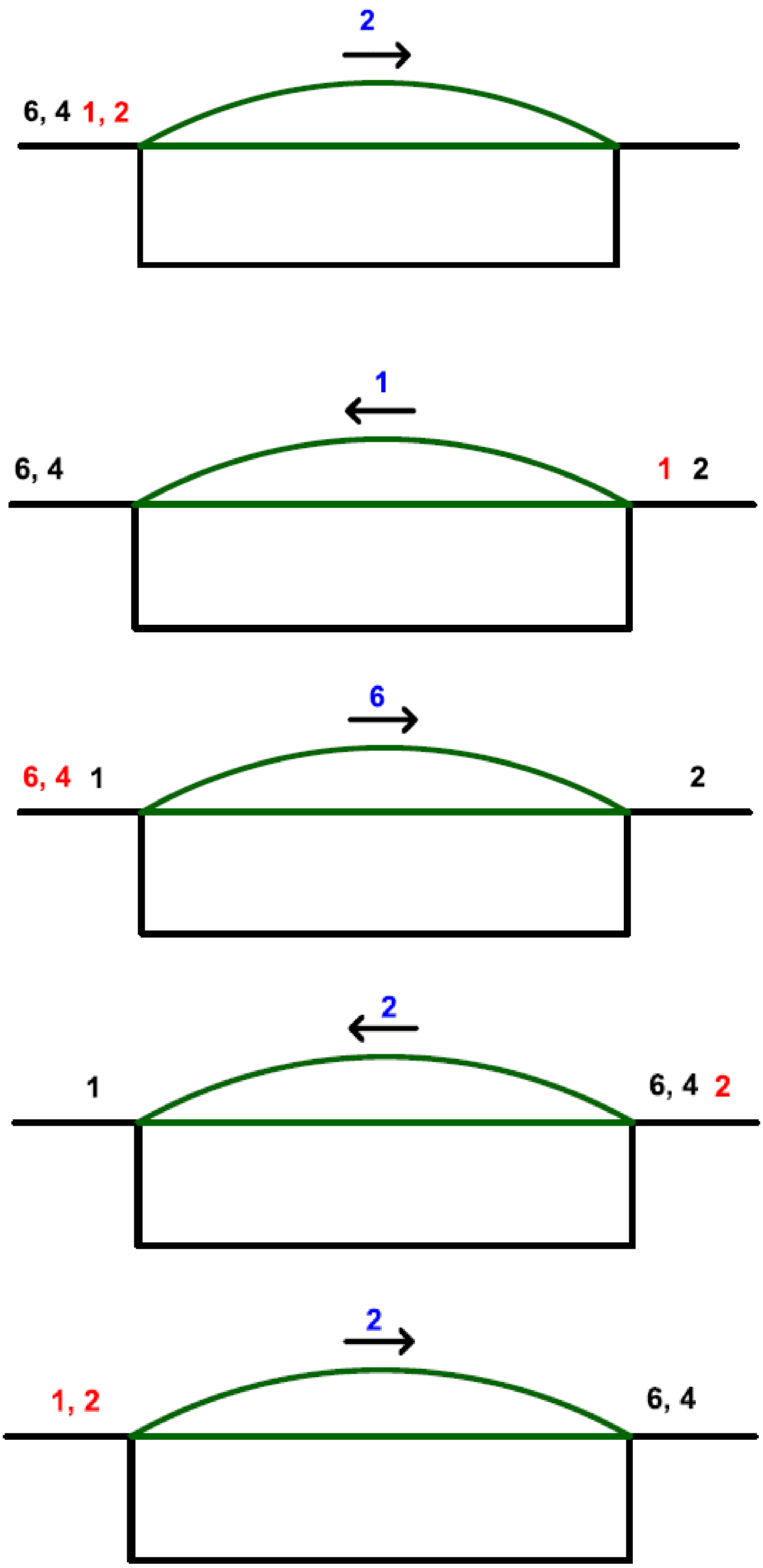
**November 9:** Find the natural numbers such that all of their divisors, except 1, are even.

**Solution:** For numbers with all their divisors except 1, even we have that their divisors have to be powers of two. Then, in the factorial decomposition of the number, the factor 2 must appear. If there were another prime in the factorial decomposition of the number, for example, p, p would be a non-even divisor of the number and all its divisors would not be even. Therefore, the factor 2 must appear in the factorial decomposition of the number and no other factor can appear. In short, the numbers alluded to in the statement are the powers of 2.

**November 11-18:** Four wounded soldiers have to cross a heavily damaged bridge at night to escape the enemy. The bridge only supports the weight of two soldiers at a time. When two soldiers cross it, they must do so at the speed of the slower one. The four soldiers only have one flashlight that has to be used every time they cross the bridge.

Individually they take 1, 2, 4 and 6 minutes to cross the bridge. What is the minimum time it takes for all four to cross it?

**Solution:** The following diagram shows the solution. In it, the soldiers are represented by the minutes it takes to cross the bridge. In red, the pair of soldiers or the soldier who is going to cross to the other part of the bridge. Above the bridge, in blue, are the minutes it takes to cross the bridge.




The time for all crossings is: 2 + 1 + 6 + 2 + 2 = 13 minutes.

**<u>November 12:</u>** The participants of a TV contest have to answer 30 questions. If they are correct, they receive a point. If they fail, half a point is deducted. If they don't answer they get zero points. If a contestant received six points, detail their answers.

**Solution:** Let x (y, z) be the number of correct (incorrect, unanswered) answers. From the statement, we have:

$$\left.\begin{matrix} x+y+z=0 \\ x-\frac{y}{2}=6 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x+y+z=30 \\ 2x-y=12 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x+y+z=30 \\ y=2x-12 \end{matrix}\right\} \Rightarrow x+2x-12+z=30 \Rightarrow 3x+z=42$$

Also:

$$y=2x-12\ \geq 0 \Rightarrow x \geq 6$$

$$z=42-3x \geq 0 \Rightarrow \frac{42}{3} \geq x \Rightarrow 14 \geq x$$

Then x ∈ {6, 7, 8, 9, 10, 11, 12, 13, 14} and we will have:

| x correct | z = 42 – 3x unanswered | y = 2x – 12 incorrect |
|---|---|---|
| 6 | 24 | 0 |
| 7 | 21 | 2 |
| 8 | 18 | 4 |
| 9 | 15 | 6 |
| 10 | 12 | 8 |
| 11 | 9 | 10 |
| 12 | 6 | 12 |
| 13 | 3 | 14 |
| 14 | 0 | 16 |

**November 13:** A person has a saddle valued at €50 and two horses. If you place the chair in the first, its value is double the second. If you place the chair in the second, its value is triple the first. What is the value of each horse?

**Solution:** Let x (y) be the value of the first (second) horse. We will have, according to the statement:

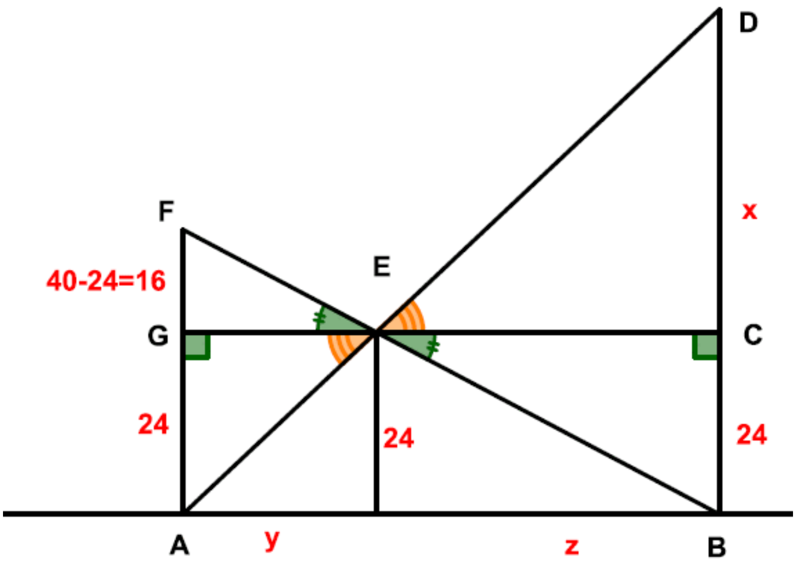
$$\left.\begin{matrix} x+50=2y \\ 3x=y+50 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x=2y-50 \\ 3x=y+50 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 3x=6y-150 \\ 3x=y+50 \end{matrix}\right\} \Rightarrow 6y-150=y+50 \Rightarrow y=40$$

$$x=2\cdot 40-50=30$$

That is, the first horse has a value of €30 and the second horse a value of €40

**November 15-22:** Each of two vertical poles of different heights placed on flat ground has a device on its top that directs a laser beam at the base of the other pole. If the rays intersect at a height of 24 m from the ground and if the shortest of the sticks has a height of 40 m, what is the height of the tallest stick?

**Solution:** Let's draw the described situation:




Since the coloured angles in E are opposite at the vertex, they will be equal. So:

$$\Delta GAE \approx \Delta EDC \Rightarrow \frac{24}{x} = \frac{y}{z}$$

$$\Delta FGE \approx \Delta ECB \Rightarrow \frac{24}{16} = \frac{z}{y}$$

So:

$$\frac{24}{x} = \frac{16}{24} \Rightarrow x = \frac{24^2}{16} = 36$$

Then the tallest stick measures (36 + 24 =) 60 meters

**November 16-17:** I have two coins. One has a 7 on one face and the other has a 10. If we toss the two coins in the air and add the results that appear on their upper faces, we obtain: 11, 12, 16 and 17. What numbers can the numbers of the two coins?

**Solution:** We will assume that the first coin is (7|x) and the second coin is (10|y). From the possible results when tossing the two coins we will have that the sets {7 + y, x + 10, x + y} and {11, 12, 16} must be equal. That is: 7 + and it can be any of the results: 11, 12 or 16; x + 10 will be whichever of the two remain and finally x + y will be the last result that remains. There are in total (3·2·1 =) 6 possible systems. Let's take a look at each of these possibilities.:

$$\left.\begin{array}{r} 7+y=11 \\ x+10=12 \\ x+y=16 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=11-7=4 \\ x=12-10=2 \\ x+y=4+2=6\neq 16 \end{array}\right\} \Rightarrow \text{It's not possible}$$

$$\left.\begin{array}{r} 7+y=11 \\ x+10=16 \\ x+y=12 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=11-7=4 \\ x=16-10=6 \\ x+y=4+6=10\neq 12 \end{array}\right\} \Rightarrow \text{It's not possible}$$

$$\left.\begin{array}{r} 7+y=12 \\ x+10=11 \\ x+y=16 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=12-7=5 \\ x=11-10=1 \\ x+y=5+1=6\neq 16 \end{array}\right\} \Rightarrow \text{It's not possible}$$

$$\left.\begin{array}{r} 7+y=12 \\ x+10=16 \\ x+y=11 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=12-7=5 \\ x=16-10=6 \\ x+y=5+6=11 \end{array}\right\} \Rightarrow \textbf{Solution: (7|6) and (10|5)}$$

$$\left.\begin{array}{r} 7+y=16 \\ x+10=11 \\ x+y=12 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=16-7=9 \\ x=11-10=1 \\ x+y=1+9=10\neq 12 \end{array}\right\} \Rightarrow \text{It's not possible}$$

$$\left.\begin{array}{r} 7+y=16 \\ x+10=12 \\ x+y=11 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=16-7=9 \\ x=12-10=2 \\ x+y=2+9=11 \end{array}\right\} \Rightarrow \textbf{Solution: (7|2) and (10|9)}$$

**November 19-20:** Two elevators leave, empty, from the sixth floor of a building at two in the afternoon and both go down. The fastest takes one minute to go from one floor to another, while the slowest takes two minutes. The first elevator to reach a floor will have to stop for three minutes to open doors, load and unload passengers, and close doors. Which elevator reaches the first floor first? At what time will each elevator arrive at the ground floor?

**Solution:** We will make a detailed tour of the descent of the two elevators:

| 0m | Fast | Slow |
|---|---|---|
| 1m | Get to P5. Stop | |
| 2m | | |
| 3m | | |
| 4m | Closed doors in P5. Come down | Get to P4. Stop |
| 5m | | |
| 6m | P3 arrives. Stop | |
| 7m | | Closed doors in P4. Come down |
| 8m | | |
| 9m | Closed doors in P3. Come down | Get to P3. Not stop |
| 10m | Get to P2. Stop | |
| 11m | | Get to P2. Not stop |
| 12m | | |
| 13m | Closed doors in P2. Come down | Get to P1. Stop |
| 14m | Get to P1. Not stop | |
| 15m | Arrive at P0. Stop | |
| 16m | | Closed doors in P1. Come down |
| 17m | | |
| 18m | | Arrive at P0. Stop |

The slow elevator reaches the first floor first at 2:13 p.m. The fast one arrives at the ground floor at 2:15 p.m. and the slow one at 2:18 p.m.

**November 23:** Find the natural numbers such that half of their divisors are even and the other half are odd.

**Solution:** Let N be one of such numbers. Then, in the factorial decomposition of N as a product of prime numbers, the factor 2 raised to 1 must appear. For these numbers (N = 2 m) we will have m that does not contain the factor 2. Each divisor of m: q generates two divisors of N, the divisor of m itself: q and the divisor of N: 2·q. Since there is no factor 2 in m, all its divisors are odd. Therefore, the divisors of N will be half odd and half even.

**November 24:** Find the pairs of four-digit palindromes whose sum is a five-digit palindrome.

**Solution:** We will have:

$$\begin{array}{ccccc} & a & b & b & a \\ + & c & d & d & c \\ \hline x & y & z & y & x \end{array}$$

1.- Let's look at the digit x of the hundreds of thousands of the sum: Since this is the sum of two other digits, we have that:

$$a \leq 9, c \leq 9 \ \Rightarrow c + a \leq 18 + 1 = 19 \ \Rightarrow x = 1$$

(because possibly we can "carry" 1 of the previous sum).

2.- We will have:

$$\begin{array}{ccccc} & a & b & b & a \\ + & c & d & d & c \\ \hline 1 & y & z & y & 1 \end{array}$$

Let us now notice that the sum of thousand units (of the four-digit palindrome) plus those that we possibly "carry" (1) gives 10 + y. Namely:

$$a + c + (1) = 10 + y \ \ \Rightarrow \ \ a + c = 10 + y - (1)$$

(where (1) indicates that "possibly we carry" 1). That is, a + c ends in y – (1). And now, in the sum of units of the two four-digit palindrome: a + c ends in 1. Then:

$$y - (1) = 1 \quad \Rightarrow \quad \begin{matrix} y = 1 \\ y = 2 \end{matrix}$$

That is: a + c = 11 and also, y = 1 or y = 2

3.- suppose y =1

$$\begin{array}{ccccc} & & & 1 & \\ & a & b & b & a \\ + & c & d & d & c \\ \hline 1 & 1 & z & 1 & 1 \end{array}$$

So by adding the tens of the four-digit palindromes:

$$1 + b + d = 1 \ (*) \ \Rightarrow \ \ b + d = 0 \ \ \Rightarrow \ \ b = d = 0 \ \ \Rightarrow z = 0$$

And one possible result is:

$$\begin{array}{ccccc} & & & 1 & \\ & a & 0 & 0 & a \\ + & c & 0 & 0 & c \\ \hline 1 & 1 & 0 & 1 & 1 \end{array}$$

and since a + c = 11:

| a | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|
| c | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

We get eight cases

(*) The case 1 + b + d = 10 + 1, that is b + d = 10, leads to a five-digit number that is not a palindrome:

$$
\begin{array}{ccccc}
 & 1 & 1 & 1 & \\
 & a & b & b & a \\
+ & c & d & d & c \\
\hline
1 & 2 & 1 & 1 & 1
\end{array}
$$

4.- suppose y = 2

$$
\begin{array}{ccccc}
 & & & 1 & \\
 & a & b & b & a \\
+ & c & d & d & c \\
\hline
1 & 2 & z & 2 & 1
\end{array}
$$

So by adding the tens and hundreds of four-digit palindromes:

$$
\left.\begin{array}{l} b + d + 1 = 12\ (**) \\ b + d + 1 = 10 + z \end{array}\right\} \Rightarrow\ b + d = 11 \text{ y } z = 2
$$

Then other results are:

$$
\begin{array}{ccccc}
 & 1 & 1 & 1 & \\
 & a & b & b & a \\
+ & c & d & d & c \\
\hline
1 & 2 & 2 & 2 & 1
\end{array}
$$

Where a + c = 11:

| a | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|
| c | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

and b + d = 11

| b | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|
| d | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

(**) The case 1 + b + d = 2, that is b + d = 1, leads to a five-digit number that is not a palindrome:

$$
\begin{array}{ccccc}
 & & & 1 & \\
 & a & b & b & a \\
+ & c & d & d & c \\
\hline
1 & 1 & 1 & 2 & 1
\end{array}
$$

In total, each pair of figures (a, c), with a + c = 11, originates (8 + 1 =) 9 cases for the pair (b, d):

| b | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| d | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

There are in total (89 =) 72 possible cases.

**<u>November 25-26:</u>** At home I have an alarm clock that goes back two minutes every hour, while my wristwatch goes one minute forward every hour. One day I adjust the time of both clocks and leave the house. When I returned the clock on my wrist shows 12 at night and instead, on the alarm clock it was 11 at night. How long was I away from home? What was the exact time when I entered the house?

**Solution:** Every hour that passes there is a discrepancy of three minutes between the two clocks. As when returning home, the discrepancy between them is 60 minutes, they have spent

$$\left(\frac{60}{3}=\right)\ \mathbf{20\ horas}$$

When I enter the house, my watch (which advances one minute per hour) shows 12:00 p.m. As it takes a 20-minute advance, they are exactly the **23.40**

**November 27:** The product of a two-digit number by its own digits gives 1950. Find this number.

**Solution:** If 10 a + b is the desired number; we will have:

$$1950=\begin{cases}=2\cdot 3\cdot 5^2\cdot 13\\=(10a+b)\cdot a\cdot b\end{cases}$$

We will finish the problem by writing the prime factors of 1950 as a product of three factors, two of them with one digit and another with two digits that are the unit factors.

By simple manipulation of the prime factors:

$$1950=\begin{cases}=6\cdot 5\cdot(5\cdot 13)=6\cdot 5\cdot 65\\=(10a+b)\cdot a\cdot b\end{cases}$$

Then there is a unique number that satisfies the statement: 65.

**November 29-30:** Place 1, or –1, in each box of a 4x4 grid so that the product of all the numbers in a row or column is always –1. What is the minimum and maximum amount of –1 that we should put? What would these quantities be on an nxn grid?

**Solution:** If the product of the numbers in each row or column is to give –1, then there must be at least one –1 in each row and column. And this is possible by putting a –1 in each square of the main or secondary diagonal. It is also possible in many other ways

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | **-1** | -1 |
| 1 | 1 | **-1** | 1 | -1 |
| 1 | **-1** | 1 | 1 | -1 |
| **-1** | 1 | 1 | 1 | -1 |
| -1 | -1 | -1 | -1 | |

| | | | | |
|---|---|---|---|---|
| 1 | 1 | **-1** | 1 | -1 |
| 1 | 1 | 1 | **-1** | -1 |
| 1 | **-1** | 1 | 1 | -1 |
| **-1** | 1 | 1 | 1 | -1 |
| -1 | -1 | -1 | -1 | |

| | | | | |
|---|---|---|---|---|
| 1 | 1 | **-1** | 1 | -1 |
| 1 | 1 | 1 | **-1** | -1 |
| **-1** | 1 | 1 | 1 | -1 |
| 1 | **-1** | 1 | 1 | -1 |
| -1 | -1 | -1 | -1 | |

| | | | | |
|---|---|---|---|---|
| **-1** | 1 | 1 | 1 | -1 |
| 1 | 1 | **-1** | 1 | -1 |
| 1 | 1 | 1 | **-1** | -1 |
| 1 | **-1** | 1 | 1 | -1 |
| -1 | -1 | -1 | -1 | |

The maximum number of –1 comes out by putting a total of three –1 and one 1 in each row or column. And this is achieved by changing in each of the solutions of the previous section, each 1 by –1 and each –1 by 1. For example, for the first solution of the previous section we will have the attached figure.

A maximum of (4·4 – 4 =) 12 numbers –1 can be placed

| | | | | |
|---|---|---|---|---|
| **-1** | **-1** | **-1** | 1 | -1 |
| **-1** | **-1** | 1 | **-1** | -1 |
| **-1** | 1 | **-1** | **-1** | -1 |
| 1 | **-1** | **-1** | **-1** | -1 |
| -1 | -1 | -1 | -1 | |

If the grid has dimension nxn, the minimum number of –1s that we can place is n (for example, placing a –1 in each box of the main or secondary diagonal).

The maximum number of cells with –1, is:

$$\mathrm{n\ odd} \;\Rightarrow\; \mathrm{n}^2 \;\; (\text{all boxes can have} -1)$$

$$\mathrm{n\ even} \Rightarrow \begin{cases} \text{in all boxes except one in each} \\ \text{row or each column, that is, except} \\ \text{in n cells} \Rightarrow \mathrm{n}^2 - \mathrm{n} = \mathrm{n} \cdot (\mathrm{n} - 1) \end{cases}$$