SOLUTIONS JANUARY 2022

PROBLEMS FOR HIGH SCHOOL. AUTHORS: Collective "CONCURSO DE PRIMAVERA"

http://www.concursoprimavera.es/#concurso.

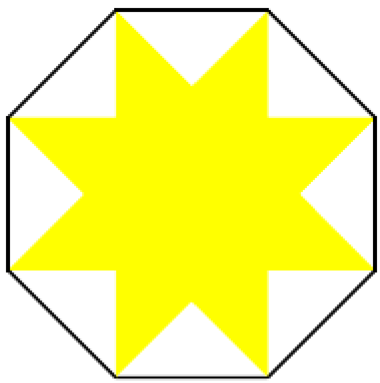
January 1: How many numbers less than 100 are the product of three prime numbers?

Solution: The largest prime factor that any of the numbers sought can have is 23, since the smallest number product of three prime factors with a factor greater than 23 is: 2 2 29 = 116. Thus, it is necessary to see how many numbers less than 100 are products of three factors from the list: 2, 3, 5, 7, 11, 13, 17, 19, 23. The best way to obtain them is to write, in order, three factors so that each factor is greater than or equal to that the factor to its left:

2·2·2; 2·2·3; 2·2·5; 2·2·7; 2·2·11; 2·2·13; 2·2·17; 2·2·19; 2·2·23	(9 numbers)
2·3·3; 2·3·5; 2·3·7; 2·3·11; 2·3·13	(5 numbers)
2·5·5; 2·5·7; 2·7·7	(3 numbers)
3·3·3; 3·3·5; 3·3·7; 3·3·11; 3·5·5	(5 numbers)

In total, (9 +5 + 3 + 5 =) 22 numbers

January 3: In the figure there is a regular octagon of side 4 cm. Find the area of the octagonal star

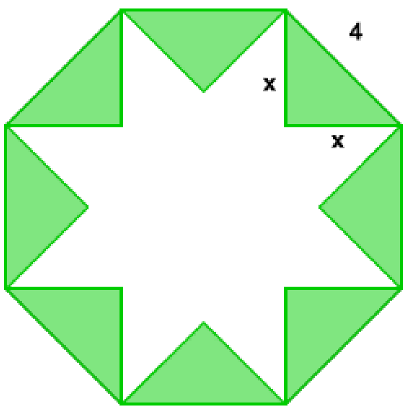


Solution: To calculate the area of the star, we will subtract the area of the eight green isosceles right triangles in the illustration to the left from the area of the octagon. Applying Pythagoras, the legs of these triangles measure:

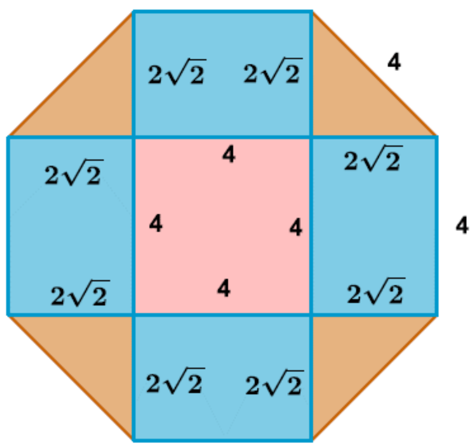
$$2x^2 = 16 \ \Rightarrow \ x = 2\sqrt{2}$$

Therefore, from the area of the octagon we must subtract:

$$8 \cdot \frac{2\sqrt{2} \cdot 2\sqrt{2}}{2} = 32$$



To find the area of the octagon, we will calculate the area of the red square, the area of the 4 blue rectangles and the area of the 4 brown triangles (see figure on the right). We will have:



$$A = 4^2 + 4 \cdot (2\sqrt{2} \cdot 4) + 4 \cdot \frac{2\sqrt{2} \cdot 2\sqrt{2}}{2} - 32 = 32 + 32\sqrt{2} - 32 = 32\sqrt{2}$$

January 4: A bag contains m white balls and n black balls. We draw a ball at random and return it by adding k balls of the same colour as the one drawn. We draw another ball at random, what is the probability that the second ball drawn is white?

Solution: Let us consider the events:

B_1 = "the first ball drawn is white"

B_2 = "the second ball drawn is white"

N_1 = "the first ball drawn is black"

Then:

$$P(B_2) = P[(B_1 \cap B_2) \cup (N_1 \cap B_2)] = P(B_1 \cap B_2) + P(N_1 \cap B_2) = \frac{m}{m+n} \cdot \frac{m+k}{m+n+k} + \frac{n}{m+n} \cdot \frac{m}{m+n+k}$$
$$= \frac{m(m+n+k)}{(m+n)\cdot(m+n+k)} = \frac{m}{m+n}$$

January 5: How much is the sum of all the products of two different natural numbers taken from 1 to n?

Previous lemma:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof of the lemma: We assume that:

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1$$

Giving different values to k, we have:

$$k = 1 \quad \Rightarrow \quad 2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$
$$k = 2 \quad \Rightarrow \quad 3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$
$$k = 3 \quad \Rightarrow \quad 4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$
$$\vdots \qquad\qquad \vdots$$
$$k = n \quad \Rightarrow \quad (n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

Adding member to member all the equalities, we will have:

$$(n+1)^3 - 1 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

From where:

$$3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - \frac{3}{2}n(n+1) - n = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{2} = \frac{(n+1)n(2n+1)}{3}$$

With which we arrive at the thesis of the lemma.

Solution: We have:

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^2 + 2S$$

Where S is the sum of all the products of two different natural numbers taken from 1 to n. From the previous equality, we have:

$$\left(\frac{n(n+1)}{2}\right)^2 = \sum_{k=1}^{n} k^2 + 2S$$

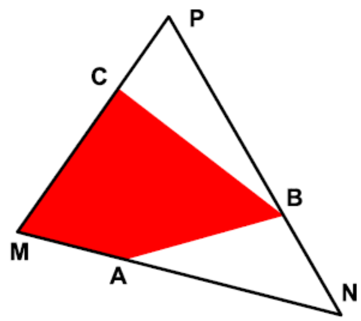
Applying the motto:

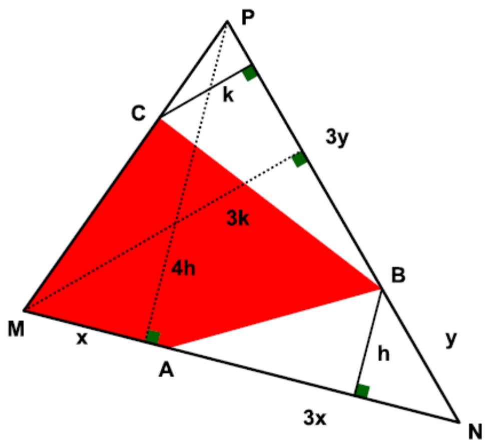
$$\left(\frac{n(n+1)}{2}\right)^2 = \sum_{k=1}^{n} k^2 + 2S = \frac{n(n+1)(2n+1)}{6} + 2S$$

So:

$$S = \frac{\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}}{2} = \frac{n(n+1)(n-1)(3n+2)}{24}$$

January 6-7: Points A, B and C in the figure divide each side of the triangle ΔMNP into two pieces that are in the ratio 1:3. Find the ratio between the area of the region coloured red and the area of the triangle ΔMNP





Solution: We have:

$$\left.\begin{array}{l} A_{\Delta ABN} = \dfrac{3xh}{2} \\ A_{\Delta MNP} = \dfrac{4x \cdot 4h}{2} \end{array}\right\} \Rightarrow \frac{A_{\Delta ABN}}{A_{\Delta MNP}} = \frac{3}{16}$$

$$\left.\begin{array}{l} A_{\Delta CBP} = \dfrac{3yk}{2} \\ A_{\Delta MNP} = \dfrac{4y \cdot 4k}{2} \end{array}\right\} \Rightarrow \frac{A_{\Delta ABN}}{A_{\Delta MNP}} = \frac{3}{16}$$

then:

$$\frac{A_{MABC}}{A_{\Delta MNP}} = 1 - \frac{3}{16} - \frac{3}{16} = \frac{5}{8}$$

January 8: For what values of x does the following expression reach its largest value and what is this?

$$\mathbf{\frac{\sin^3 x \cdot \cos x}{1 + tag^2 x}}$$

Solution: We have:

$$f(x) = \frac{\sin^3 x \cdot \cos x}{1 + tag^2 x} = \frac{\sin^3 x \cdot \cos x}{\frac{1}{\cos^2 x}} = \sin^3 x \cdot \cos^3 x = \frac{1}{8}(2\sin x\cos x)^3 = \frac{1}{8}(\sin 2x)^3$$

Since the function y = sin x oscillates between –1 and 1 and the function $y = x^3$ is strictly increasing, we will have that the maximum of the function is reached when:

$$\sin 2x = 1 \ \text{ i.e. } \ x = 45° \pm k \cdot 360°$$

and in this case, the maximum of the function is 1/8.

January 10: If

$$\mathbf{x^2 + xy + y^2 = 84}$$
$$\mathbf{x - \sqrt{xy} + y = 6}$$

find x·y

Solution: The first equation can be rewritten as:

$$x^2 + 2xy + y^2 = 84 + xy \ \Rightarrow \ (x + y)^2 = 84 + xy$$

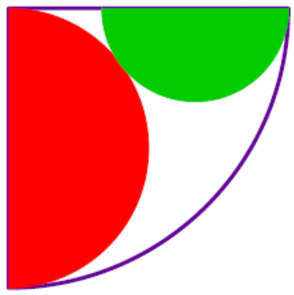
With the second equation, we have:

$$x + y = 6 + \sqrt{xy} \ \Rightarrow \ (x + y)^2 = \left(6 + \sqrt{xy}\right)^2 = 36 + xy + 12\sqrt{xy}$$

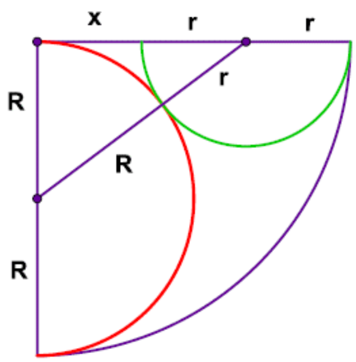
Matching the second members:

$$84 + xy = 36 + xy + 12\sqrt{xy} \ \Rightarrow 84 - 36 = 12\sqrt{xy} \ \Rightarrow xy = \left(\frac{48}{12}\right)^2 = 16$$

January 11: The drawing shows a quadrant of radius s and two tangent semicircles. Find the radius of the small semicircle.



Solution:



Let R(r) be the radius of the large (small) semicircle. Applying Pythagoras we will have:

$$\left.\begin{array}{l} R^2 + (x + r)^2 = (R + r)^2 \\ x + 2r = 2R = s \end{array}\right\}$$

Solving for x in the second equation and substituting in the first:

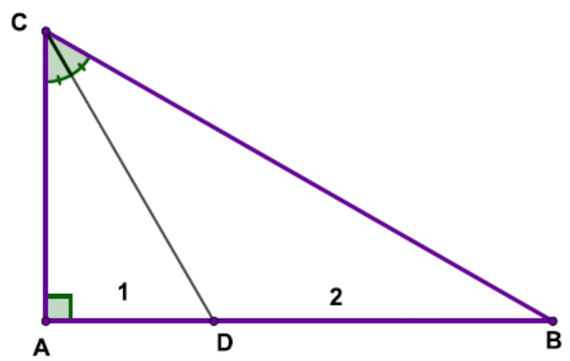
$$R^2 + (2R - 2r + r)^2 = (R + r)^2$$

which leads to:

$$4R(R - r) = 2Rr \ \Rightarrow \ r = \frac{2R}{3} = \frac{s}{3}$$

January 12: In a right triangle, the bisector of an acute angle cuts the opposite leg into two pieces of length 1 and 2: What is the length of the interior bisector segment of the triangle?

Solution:



By the bisector theorem, we have:

$$\frac{CA}{AD} = \frac{CB}{DB}$$

If a=AC, we will have:

$$\frac{a}{1} = \frac{CB}{2} \quad \Rightarrow \quad CB = 2a$$

And applying Pythagoras, in ΔABC:

$$CB^2 = AB^2 + CA^2 \quad \Rightarrow \quad 4a^2 = 9 + a^2 \quad \Rightarrow \quad a = \sqrt{3}$$

and now in ΔCAD:

$$CD^2 = 1 + a^2 = 1 + 3 = 4 \quad \Rightarrow \quad CD = 2$$

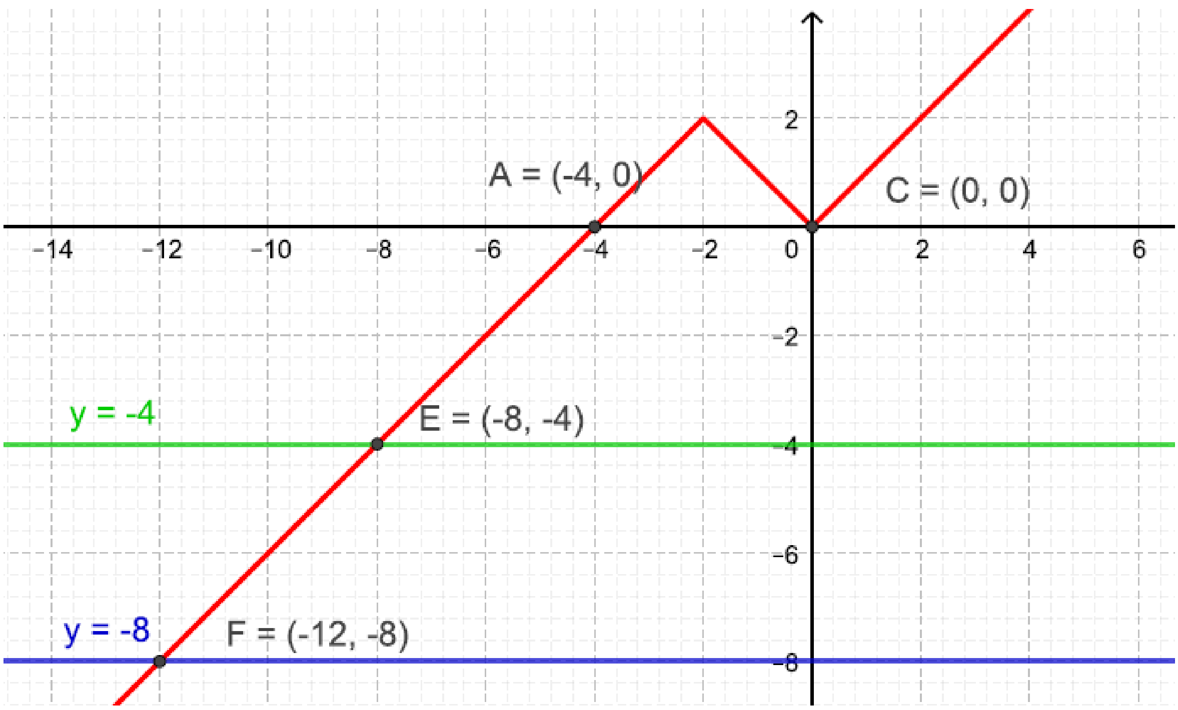
January 13: Consider the natural numbers with nine digits. How many numbers do we have to extract to ensure that at least two of them have the same digit in the ten thousand?

Solution: The figure that appears in the ten thousand can be any of the set {0, 1, 2, 3, 4, 5, 6,7, 8, 9}. If we call nest I the numbers that have the digit i in the tens of thousands, we will have a total of 10 nests. Extracting from the initial set 11 numbers we will have that at least 2 are in the same nest, that is, at least two will have the same digit in the ten thousand.

January 14: Solve **f(f(f(x))) = 0**, where:

$$\mathbf{f(x)} = \begin{cases} \mathbf{x + 4 \text{ sii } x \leq -2} \\ \mathbf{-x \text{ sii } -2 < x < 0} \\ \mathbf{x \quad \text{sii } x \geq 0} \end{cases}$$

Solution: It is enough to have the graphical representation of the function and analyse the equation provided.



We have:

$$f\left(f\left(f(x)\right)\right) = 0 \Rightarrow f\left(f(x)\right) = \begin{cases} = 0 & \Rightarrow f(x) = \begin{cases} = 0 & \Rightarrow x = \begin{cases} = 0 \\ = -4 \end{cases} \\ = -4 & \Rightarrow \quad x = -8 \end{cases} \\ = -4 & \Rightarrow \quad f(x) = -8 \quad \Rightarrow \quad x = -12 \end{cases}$$

Then the equation has four roots, namely: x = 0, x = – 4, x = – 8, x = – 12.

January 15: From the function f(x) it is known that it is periodic with period 5 and that in [3, 8[it verifies:

$$\mathbf{f(x) = x^2 - 10x + 25}$$

Find f(2022).

Solution: Let's see to which interval the value x = 2022 belongs.

$$2022 \geq 3 + 5(n-1) \ \Rightarrow \ \frac{2022 - 3}{5} \geq n - 1 \ \Rightarrow 403{,}8 \geq n - 1 \ \Rightarrow n = 404$$

Namely:

$$3 + 5(404 - 1) = 2018 \leq 2022 \leq 8 + 5(404 - 1) = 2023$$

Since 2022 – 2018 = 4, we will have:

$$f(2022) = f(3 + 4) = f(7) = 7^2 - 10 \cdot 7 + 25 = 4$$

January 17: How many pairs of integers (x, y) with x ≤ y, verify that their product is equal to five times their sum?

Solution: We have to solve in $\mathbb{Z}$ the equation:

$$x \cdot y = 5(x + y) \quad \text{con} \quad x \leq y$$

We have:

$$xy - 5y = 5x \ \Rightarrow \ y(x - 5) = 5x \ \Rightarrow \ y = \frac{5x}{x-5} = \frac{5x - 25 + 25}{x - 5} = 5 + \frac{25}{x-5}$$

(since x ≠ 5, then if x = 5 ⇒ 5y = 5(5 + y) =25 + 5y ⇒ 0 = 25). So x – 5 must be a divisor of 25, i. e.:

$$x - 5 \in \{\pm 1, \pm 5, \pm 25\}$$

Let's analyse, case by case:

$$x - 5 = 1 \ \Rightarrow \ x = 6 \ \Rightarrow y = 5 + \frac{25}{1} = 30 \ \Rightarrow \ \mathbf{x = 6;\ \ y = 30 \text{ is solution}}$$

$$x - 5 = -1 \ \Rightarrow \ x = 4 \Rightarrow y = 5 + \frac{25}{-1} = -20 \Rightarrow x = 6;\ y = -20 \text{ is not a solution because } x = 4 > y = -20$$

$$x - 5 = 5 \ \Rightarrow \ x = 10 \ \Rightarrow y = 5 + \frac{25}{5} = 10 \ \Rightarrow \ \mathbf{x = 10;\ \ y = 10 \text{ is solution}}$$

$$x - 5 = -5 \ \Rightarrow \ x = 0 \ \Rightarrow y = 5 + \frac{25}{-5} = 0 \ \Rightarrow \ \mathbf{x = 0;\ \ y = 0 \text{ is solution}}$$

$$x - 5 = 25 \Rightarrow \ x = 30 \Rightarrow y = 5 + \frac{25}{25} = 6 \ \Rightarrow \ x = 30;\ y = 6 \text{ is not a solution because } x = 30 > y = 6$$

$$x - 5 = -25 \Rightarrow x = -20 \Rightarrow y = 5 + \frac{25}{-25} = 4 \ \Rightarrow \ \mathbf{x = -20;\ \ y = 4 \text{ is solution}}$$

January 18: What is the remainder of dividing

$$\mathbf{P(x) = x^{200} - 2x^{199} + x^2 + x + 1}$$

by D(x) = x^2 - 3x + 2?

Solution 1: We will have, since D(x) = (x – 1)·(x – 2), that there are polynomials q(x) and ax +b:

$$P(x) = q(x) \cdot (x-1) \cdot (x-2) + ax + b$$

Giving x the value 1, we have:

$$P(1) = \begin{cases} = 1^{200} - 2 \cdot 1^{199} + 1^2 + 1 + 1 = 2 \\ = q(1) \cdot (1-1) \cdot (1-2) + a + b \end{cases} \Rightarrow 2 = a + b$$

Giving x the value 2, we have:

$$P(2) = \begin{cases} = 2^{200} - 2 \cdot 2^{199} + 2^2 + 2 + 1 = 7 \\ = q(1) \cdot (2-1) \cdot (2-2) + 2a + b \end{cases} \Rightarrow 7 = 2a + b$$

Finally:

$$\begin{cases} a + b = 2 \\ 2a + b = 7 \end{cases} \Rightarrow \quad a = 5, b = -3 \quad \Rightarrow \quad \mathbf{r(x) = 5x - 3}$$

Solution 2: Since D(x) = (x – 1) (x – 2), by the Remainder Theorem:

$$P(x) = Q_1(x)(x-1) + P(1) \quad \Rightarrow \quad (x-2)P(x) = Q_1(x)(x-1)(x-2) + P(1)(x-2)$$
$$P(x) = Q_2(x)(x-2) + P(2) \quad \Rightarrow \quad (x-1)P(x) = Q_1(x)(x-1)(x-2) + P(2)(x-1)$$

Subtracting the last two equalities:

$$\big((x-2)-(x-1)\big)P(x) = (-1)P(x) = \big(Q_1(x) - Q_2(x)\big)(x-1)(x-2) + P(1)(x-2) - P(2)(x-1)$$

From where:

$$P(x) = \big(Q_2(x) - Q_1(x)\big)(x-1)(x-2) + P(2)(x-1) - P(1)(x-2)$$

And, by the uniqueness of the decomposition of the division of polynomials, (since $\mathrm{grd}\big(P(2)(x-1) - P(1)(x-2)\big) = 1 < 2 = \mathrm{grd}\big((x-1)(x-2)\big)$, the quotient and the remainder of the division between P(x) and D(x) are:

$$C(x) = Q_2(x) - Q_1(x); \;\; R(x) = P(2)(x-1) - P(1)(x-2)$$

Finally, since P(2) = 7 and P(1) = 2, we have:

$$R(x) = 7(x-1) - 2(x-2) = 5x - 3$$

January 19: There are ten consecutive naturals. The sum of nine of them gives 2022. What number have we not added?

Solution: Let be the ten consecutive natural:

$$x-4, \qquad x-3, \qquad x-2, \qquad x-1, \qquad x, \qquad x+1, \qquad x+2, \qquad x+3, \qquad x+4, \qquad x+5$$

We will have, if the natural number, not added, is x – k (k ∈ {– 5, – 4, – 3, – 2, – 1, 0, 1, 2, 3, 4}) :

$$10x + 5 = 2022 + (x - k) \;\Rightarrow\; 9x = 2017 - k \;\Rightarrow\; x = \frac{2017 - k}{9} = \frac{224 \cdot 9 + 1 - k}{9} = 224 + \frac{1 - k}{9}$$

And since x is natural, 9 as of dividing 1 – k. The only possibility (because of the values of k) is that 1 – k = 0, that is, k = 1. Thus x = 224 and the number not added is (x – 1 =) 223

January 20: Points A and B are points on the graph of

$$y = x^2 - 7x - 1$$

Find the length of segment AB if (0, 0) is its midpoint.

Solution: Since A and B are points of the parabola, we have:

$$A(a;\ a^2 - 7a - 1)\quad B(b;\ b^2 - 7b - 1)$$

And since (0; 0) is the midpoint of the segment AB, we will have:

$$\begin{cases} \dfrac{a+b}{2} = 0 \\ \dfrac{a^2 - 7a - 1 + b^2 - 7b - 1}{2} = 0 \end{cases}$$

From the first equation:

$$b = -a$$

which substituted in the second leads to:

$$2a^2 - 2 = 0 \quad \Rightarrow \quad a = \pm 1 \quad \Rightarrow \quad b = \mp 1$$

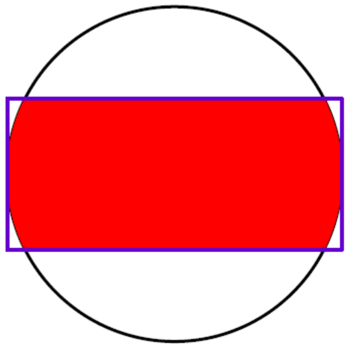
The points are then:

$$A(1;\ -7)\ \text{ y } B(-1; 7) \quad \text{o} \quad A(-1;\ 7)\ \text{y}\ \ B(1;\ -7)$$

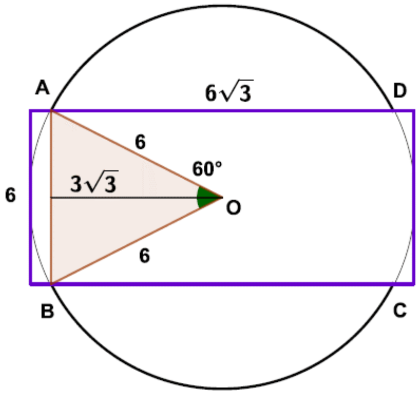
The distance between them is:

$$d(A; B) = \sqrt{(1+1)^2 + (-7-7)^2} = 10\sqrt{2}$$

January 21-22: The circle and the rectangle in the figure have the same center. The dimensions of the rectangle are 6x12 and the small sides of the rectangle are tangent to the circle, what is the area of the region common to the circle and the rectangle?



Solution:



Triangle ΔAOB is equilateral with side 6, so the angle at O is 60° and its height is

$$\sqrt{6^2 - 3^2} = 3\sqrt{3} \quad \Rightarrow \quad AD = 6\sqrt{3}$$

The area of the requested area will be the area of two circular sectors with an angle of 60° and a radius of 6 plus the area of two triangles ΔAOD.

The area of the circular sector is:

$$\frac{1}{6}\pi 6^2 = 6\pi$$

The area of the triangle ΔAOD is:

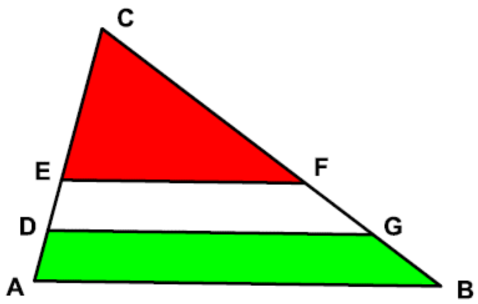
$$\frac{1}{2}\cdot 6\sqrt{3}\cdot 3 = 9\sqrt{3}$$

So the required area is:

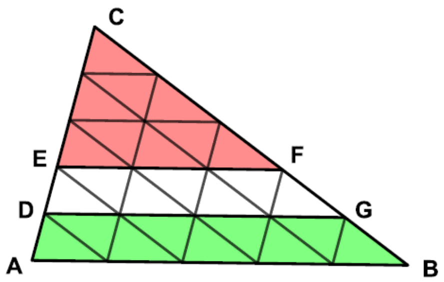
$$2\left(6\pi + 9\sqrt{3}\right) = 12\pi + 18\sqrt{3}$$

January 24-25-31: In drawing EF//DG//AB. The shaded areas have the same area and CD = 4 DA.

Find the ratio between CE and EA



Solution "wordless":



$$A_{\Delta ECF} = 9\Delta$$

$$A_{ABGD} = 9\Delta$$

$$\frac{CE}{EA} = \frac{3}{2}$$

Solution 2: Taking AD = 1 then DC= 4 and AC = 5. The triangles ΔCDG and ΔCAB are similar (because they are in Thales position) with similarity ratio

$$k = \frac{DC}{AC} = \frac{4}{5}$$

Therefore, their areas will be in proportion:

$$k^2 = \frac{16}{25}$$

So:

$$A_{\Delta CDG} = \frac{16}{25}A_{\Delta ABC}$$

$$A_{ABGD} = \frac{9}{25}A_{\Delta CEF}$$

The triangles ΔCEF and ΔCAB are also similar and the ratio of their areas is 9/25, so their similarity ratio is 3/5. From here:

$$\frac{CE}{CA} = \frac{3}{5}$$

and since CA = 5, then CE = 3 and EA = 2, whence:

$$\frac{CE}{EA} = \frac{3}{2}$$

<u>January 26:</u> Solve in $\mathbb{R}$

$$\mathbf{x^2 + y^2 = |x| + |y|}$$

<u>Solution:</u> The solutions of the equation will be the points on the graph of the expression.

Let x ≥ 0 and y ≥ 0. Then:

$$x^2 + y^2 = |x| + |y| \Rightarrow x^2 - x + y^2 - y = 0 \Rightarrow x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2} \Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

what is a circle with center $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $r = \frac{1}{\sqrt{2}}$. Naturally, only the points of this circumference that meet x ≥ 0 and y ≥ 0.

Let x ≥ 0 and y < 0. Then:

$$x^2 + y^2 = |x| + |y| \Rightarrow x^2 - x + y^2 + y = 0 \Rightarrow x^2 - x + \frac{1}{4} + y^2 + y + \frac{1}{4} = \frac{1}{2} \Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

what is a circle with center $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius $r = \frac{1}{\sqrt{2}}$. Naturally, only the points of this circumference that meet x ≥ 0 and y < 0.

Let x < 0 and y ≥ 0. Then:

$$x^2 + y^2 = |x| + |y| \Rightarrow x^2 + x + y^2 - y = 0 \Rightarrow x^2 + x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2} \Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

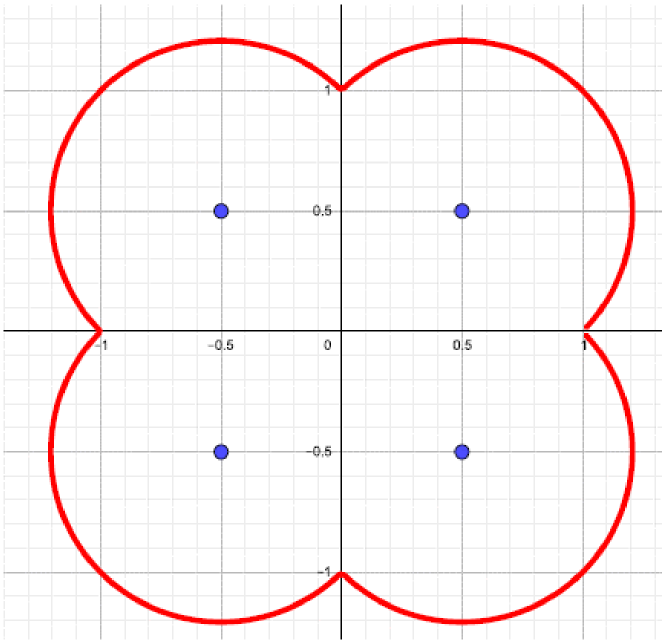
what is a circle with center $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and radius $r = \frac{1}{\sqrt{2}}$. Naturally, only the points of this circumference that meet x < 0 and y ≥ 0.

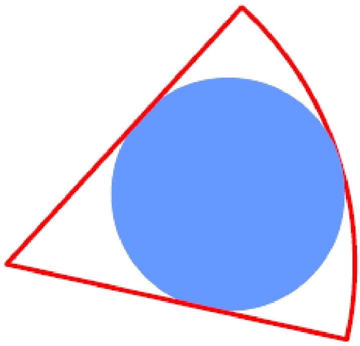
Let x < 0 and y < 0. Then:

$$x^2 + y^2 = |x| + |y| \Rightarrow x^2 + x + y^2 + y = 0 \Rightarrow x^2 + x + \frac{1}{4} + y^2 + y + \frac{1}{4} = \frac{1}{2} \Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

what is a circle with center $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and radius $r = \frac{1}{\sqrt{2}}$. Naturally, only the points of this circumference that meet x < 0 and y < 0.

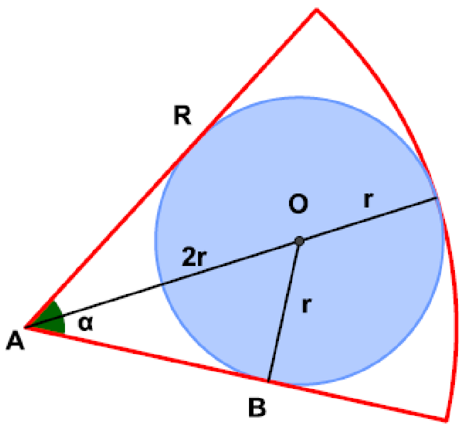
The solutions of the equation are the points on the red curve:





January 27: If the ratio of the radius of the circular sector to the radius of the circle is three, what is the ratio of their areas?

Solution:

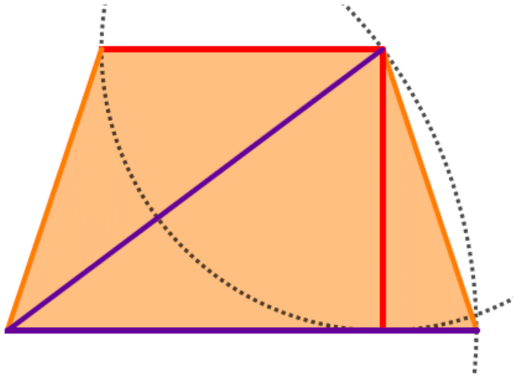


Let R(r) be the radius of the circular sector and the radius of the circle. From the statement R = 3r. Let O be the center of the circle, we will have in the right triangle ΔAOB:

$$OB = r \Rightarrow AO = 2r \Rightarrow \Delta AOB \text{ es } 30° - 60° - 90° \Rightarrow \alpha = 60°$$

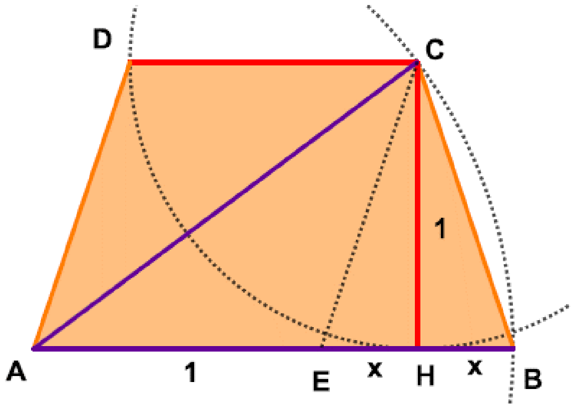
Then, if A_R (A_r) is the area of the circular sector (circle) we will have:

$$\frac{A_R}{A_r} = \frac{\frac{\pi R^2}{6}}{\pi r^2} = \frac{R^2}{6r^2} = \frac{(3r)^2}{6r^2} = \frac{3}{2}$$



January 28: If the larger base of an isosceles trapezoid measures the same as the diagonal and the smaller base measures the same as the height of the trapezoid, find the ratio between the length of the smaller base and the length of the larger base.

Solution:

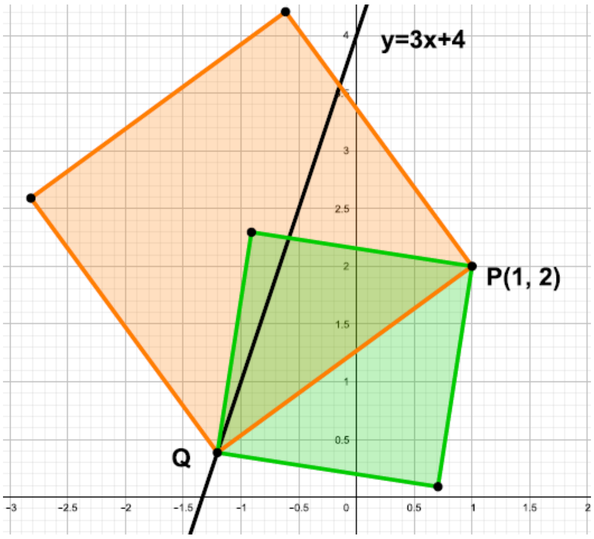


We draw segment CE, parallel to AD. Let us take the height of the trapezoid as 1. Then: AE = 1, AC = AB = 1 + 2x where x = EH = HB, and applying Pythagoras in ΔAHC, rectangle in H:

$$(1+2x)^2 = 1^2 + (1+x)^2 \;\Rightarrow\; \begin{cases} x = -1 \text{nonsense} \\ x = \dfrac{1}{3} \end{cases}$$

Therefore:

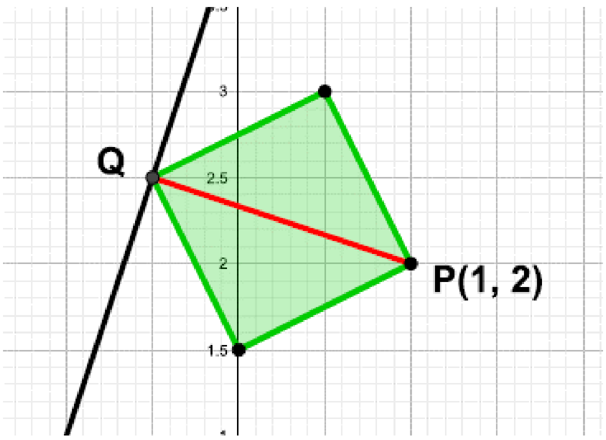
$$AB = 1 + \frac{2}{3}, \quad \frac{AB}{DC} = \frac{\frac{5}{3}}{1} = \frac{5}{3}$$



January 29: A square has one vertex at the point P(1, 2) and another on the line y = 3x + 4. What is the smallest possible value for its area?

Solution: As can be seen in the illustration on the right, choosing a point on the line creates two squares: the one with side PQ (orange) and the one with diagonal PQ (green). Since we are interested in the square with the smallest area, we will consider only the one with diagonal PQ

We are therefore looking for the point on the line y = 3x + 4 that provides the smallest possible diagonal. That point is obviously the projection of point P on the line y = 3x + 4.



The square sought has an area of:

$$A = 2 \cdot \frac{d \cdot \frac{d}{2}}{2} = \frac{d^2}{2} = \frac{10}{8} = \frac{5}{4}$$

where d is the diagonal of the square, that is:

$$d = d\big((x_0, y_0); Ax + By + C = 0\big) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|-3 + 2 - 4|}{\sqrt{(-3)^2 + 1^2}} = \frac{\sqrt{10}}{2}$$