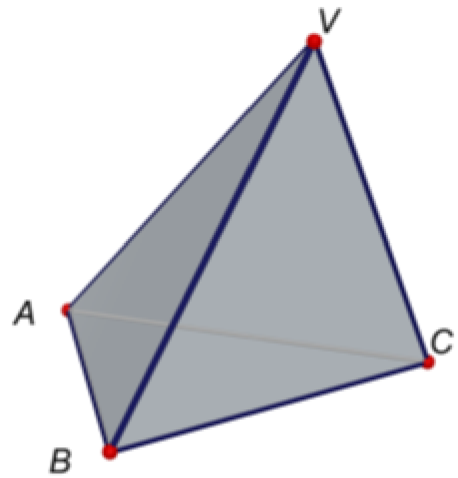
SOLUTIONS FEBRUARY 2022

PROBLEMS USING GEOMETRIC PROGRAMS. AUTHOR: RICARD PEIRÓ I ESTRUCH. IES "Abastos". València

February 1-2: The base of a tetrahedron is an equilateral triangle, and the three lateral faces unfolded and placed in one plane form a trapezoid with sides 10, 10, 10 and 14 units long.

Calculate the sum of the lengths of all the edges of the tetrahedron and also determine its area.

KöMaL C1559.

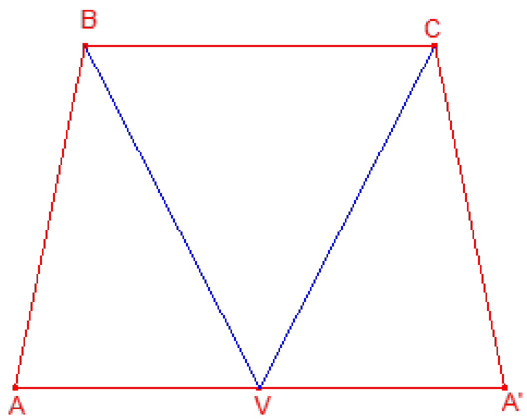


Solution: Let ABCV be the tetrahedron with the base of the equilateral triangle of sides 10, since the expansion has three sides equal to 10.
Let AA'CB be the isosceles trapezoid formed by the development of the lateral faces.
$\overline{AB}=\overline{BC}=\overline{CA'}=10$ tetrahedron base edges, $\overline{AA'}=14$
The vertex V of the tetrahedron is the midpoint of the segment $\overline{AA'}$

$$\overline{AV}=7$$

Let $\overline{BV}=\overline{CV}=a$ the other two edges. Let P be the projection of B onto $\overline{AA'}$

$$\overline{PV}=\frac{1}{2}\overline{BC}=5.$$



Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{BPV}$

$$a=11$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{APB}$

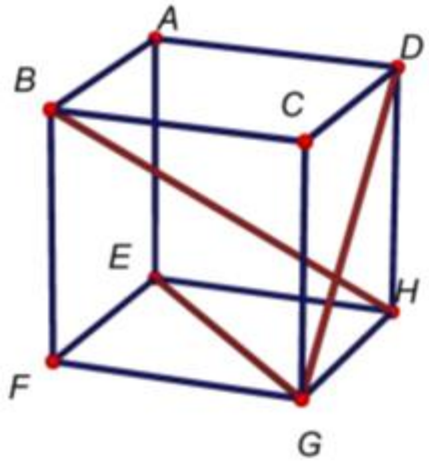
$$\overline{BP}=4\sqrt{6}$$

The sum of the lengths of the edges is:

$$L_{arestes}=3\overline{AB}+2a+\overline{AV}=3\cdot 10+2\cdot 11+7=59\ u$$

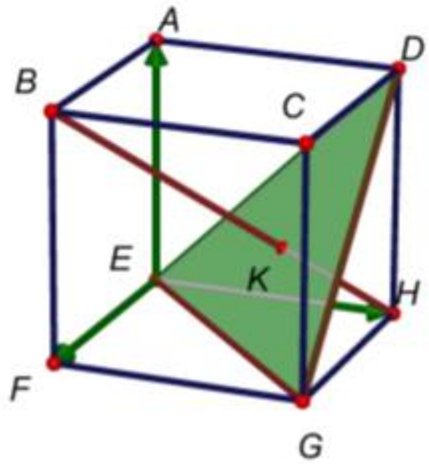
The area of the tetrahedron is:

$$S_{ABCV}=S_{ABC}+S_{AA'CB}=\frac{\sqrt{3}}{4}10^2+\frac{14+10}{2}\cdot 4\sqrt{6}=25\sqrt{3}+48\sqrt{6}$$



February 3-4: Let ABCDEFGH be the cube, edge 1.

a) Prove that $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{EG}}$.
b) Prove that $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{GD}}$.
c) Prove that $\overline{\mathrm{BH}}$ is perpendicular to plane EDG.
d) Calculate the intersection of $\overline{\mathrm{BH}}$ and the plane EDG.
e) Calculate the distance from $\overline{\mathrm{BH}}$ to plane EDG



Solution 1:

Consider the cube with the following coordinates:

$$\mathrm{E}(0,0,0), \mathrm{F}(1,0,0), \mathrm{H}(0,1,0), \mathrm{G}(1,1,0)$$
$$\mathrm{A}(0,0,1), \mathrm{B}(1,0,1), \mathrm{D}(0,1,1), \mathrm{C}(1,1,1).$$
$$\overrightarrow{\mathrm{BH}} = (-1,1,-1);\ \overrightarrow{\mathrm{EG}} = (1,1,0);\ \overrightarrow{\mathrm{GD}} = (-1,0,1);$$

A) Let us see that the vectors $\overrightarrow{\mathrm{BH}}, \overrightarrow{\mathrm{EG}}$ are orthogonal.

$$\overrightarrow{\mathrm{BH}} \cdot \overrightarrow{\mathrm{EG}} = 0.$$

So, $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{EG}}$.

B) Let us see that the vectors $\overrightarrow{\mathrm{BH}}, \overrightarrow{\mathrm{GD}}$ are orthogonal.

$$\overrightarrow{\mathrm{BH}} \cdot \overrightarrow{\mathrm{GD}} = 0.$$

So, $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{GD}}$.

C) The equation of the plane containing EDG is:

$$\Pi_{\mathrm{EDG}} \equiv \begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

Simplifying:

$$\Pi_{\mathrm{EDG}} \equiv x - y + z = 0$$

The characteristic vector of the plane is $a = (1,-1,1)$ and is the linearly dependent vector$\overrightarrow{\mathrm{BH}} = (-1,1,-1)$. So, $\overline{\mathrm{BH}}$ is perpendicular to plane EDG.

D) The equation of the line passing through B, H has equation:

$$r_{\mathrm{BH}} \equiv (x,y,z) = (1,0,1) + \alpha(-1,1,-1)$$
$$(x,y,z) = (1-\alpha, \alpha, 1-\alpha)$$

Substituting the coordinates of any point on the line in the plane:

$$(1-\alpha) - \alpha + (1-\alpha) = 0$$

Solving the equation:

$$\alpha = \frac{2}{3}$$

The intersection point of $\overline{\mathrm{BH}}$ and the plane EDG is:

$$\mathrm{K}\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

E) The distance from B to the EDG plane is equal to the distance from B to K.

$$\mathrm{d} = \sqrt{\left(\frac{1}{3} - 1\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3} - 1\right)^2} = \frac{2}{3}\sqrt{3}$$

Or, the distance from the point to the DEG plane:

$$\mathrm{d} = \left|\frac{1 - 0 + 1}{\sqrt{1^2 + (-1)^2 + 1^2}}\right| = \frac{2}{3}\sqrt{3}$$

<u>Solución 2:</u>

A) $\overline{\mathrm{BF}}$ is perpendicular to the plane AGHE. So, $\overline{\mathrm{BF}}$ is perpendicular to $\overline{\mathrm{EG}}$. $\overline{\mathrm{FH}}$ is perpendicular to $\overline{\mathrm{EG}}$, diagonals of square EGGH. So, the plane BFH is orthogonal to $\overline{\mathrm{EG}}$. So, $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{EG}}$.

B) $\overline{\mathrm{BC}}$ is perpendicular to the plane CGHD. So, $\overline{\mathrm{BC}}$ is perpendicular to $\overline{\mathrm{GD}}$. $\overline{\mathrm{CH}}$ is perpendicular to $\overline{\mathrm{GD}}$, diagonals of square CGHD. So, the plane BCH is perpendicular to $\overline{\mathrm{GD}}$. So, $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{GD}}$.

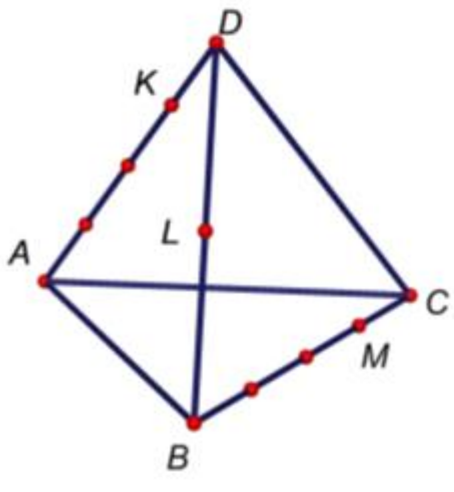
C) $\overline{\mathrm{BH}}$ is perpendicular to the plane DEG since $\overline{\mathrm{BH}}$ is perpendicular to $\overline{\mathrm{EG}}$ and $\overline{\mathrm{GD}}$.

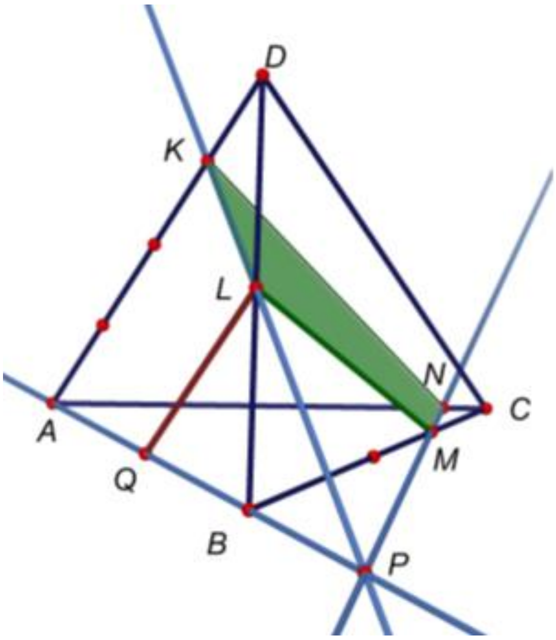
D) Note that BDEG is a regular tetrahedron since all the edges are equal to the diagonals of the faces of the cube. $\overline{\mathrm{BH}}$ is the height of the tetrahedron and its foot is K the centre of gravity of the equilateral triangle $\overset{\Delta}{\mathrm{DEG}}$.

E) $\overline{\mathrm{DK}} = \frac{2}{3}\frac{\sqrt{3}}{2}\sqrt{2} = \frac{\sqrt{6}}{3}, \overline{\mathrm{BD}} = \sqrt{2}$. Applying the Pythagorean theorem to the right triangle$\overset{\Delta}{\mathrm{BDK}}$

$$\mathrm{d} = \overline{\mathrm{BK}} = \sqrt{\left(\sqrt{2}\right)^2 - \left(\frac{\sqrt{6}}{3}\right)^2} = \frac{2}{3}\sqrt{3}$$

<u>February 5-12:</u> Let be the tetrahedron ABCD of edge 1. Let K be the point of the edge $\overline{\mathrm{AD}}$, such that $\overline{\mathrm{AK}} = 3 \cdot \overline{\mathrm{DK}}$. Let L be the midpoint of the edge $\overline{\mathrm{BD}}$. Let M be the point of the edge $\overline{\mathrm{BC}}$ such that $\overline{\mathrm{BM}} = 3 \cdot \overline{\mathrm{CM}}$. Calculate the area of the section of the tetrahedron determined by the plane passing through the points K, L, M.





Solution:

$$\overline{AK} = \overline{BM} = \frac{3}{4}, \overline{DK} = \overline{CM} = \frac{1}{4}, \overline{BL} = \overline{DL} = \frac{1}{2}$$

Lines AB, KL intersect at point P. Line PM cuts edge $\overline{AC}$ en el punto N. La section is el quadrilateral KLMN.

$$\overline{DK} = \frac{1}{4}, \overline{DL} = \frac{1}{2}, \angle KDL = 60°$$

Therefore, $\angle DKL = 90°$. So, $\overline{KL} = \frac{\sqrt{3}}{4}$

Applying the law of cosines to the triangle $\overset{\Delta}{LBM}$

$$\overline{LM}^2 = \frac{1}{4} + \frac{9}{16} - 2\frac{1}{2}\frac{3}{4}\frac{1}{2}$$

$$\overline{LM} = \frac{\sqrt{7}}{4}$$

Let Q be the midpoint of the edge $\overline{AB}$.

The triangles $\overset{\Delta}{KAP}, \overset{\Delta}{LQP}$ are similar. Applying Thales' theorem:

$$\frac{\frac{3}{4}}{\frac{1}{2}} = \frac{1 + \overline{BP}}{\frac{1}{2} + \overline{BP}}$$

Solving the equation: $\overline{BP} = \frac{1}{2}$.

Applying the law of cosines to the triangle $\overset{\Delta}{BPM}$

$$\overline{PM}^2 = \frac{1}{4} + \frac{9}{16} + 2\frac{1}{2}\frac{3}{4}\frac{1}{2}$$

$$\overline{PM} = \frac{\sqrt{19}}{4}$$

Applying the law of cosines to the triangle$\overset{\Delta}{LBP}$

$$\overline{PL}^2 = \frac{1}{4} + \frac{1}{4} + 2\frac{1}{2}\frac{1}{2}\frac{1}{2}$$

$$\overline{LM} = \frac{\sqrt{3}}{2}$$

Let us note that $\overline{PM}^2 = \overline{LM}^2 + \overline{PL}^2$. Therefore, $\angle MLP = 90°$. Applying the Pythagorean theorem to the right triangle$\overset{\Delta}{MLP}$

$$\overline{KM} = \frac{\sqrt{10}}{4}$$

Let $\angle BMP = \alpha$. Applying the law of sines to the triangle $\overset{\Delta}{BMP}$

$$\frac{\frac{1}{2}}{\sin \alpha} = \frac{\frac{\sqrt{19}}{4}}{\sin 120°}$$

So, $\sin\alpha = \frac{\sqrt{57}}{19}, \cos\alpha = \frac{4\sqrt{19}}{19}$. Applying the law of sines to the triangle $\overset{\Delta}{MCN}$

$$\frac{\overline{MN}}{\sin 60°} = \frac{\frac{1}{4}}{\sin(60° + \alpha)} = \frac{\overline{CN}}{\sin\alpha}$$

$$\overline{MN} = \frac{\sqrt{19}}{20}, \overline{CN} = \frac{1}{10}$$

$$\overline{AN} = \frac{9}{10}$$

Applying the law of cosines to the triangle $\overset{\Delta}{AKN}$

$$\overline{KN}^2 = \frac{81}{100} + \frac{9}{16} - 2\frac{9}{10}\frac{3}{4}\frac{1}{2}$$

$$\overline{KN} = \frac{3\sqrt{31}}{20}$$

Let $\angle KNM = \beta$. Applying the law of cosines to the triangle $\overset{\Delta}{KNM}$

$$\frac{10}{16} = \frac{19}{400} + \frac{279}{400} - 2\frac{\sqrt{19}}{20}\frac{3\sqrt{31}}{20}\cos\beta$$

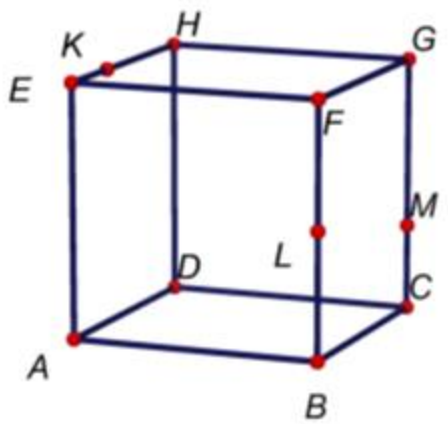
$$\cos\beta = \frac{8\sqrt{31}}{589}, \sin\beta = \frac{5\sqrt{651}}{589}$$

The area of quadrilateral KLMN is:

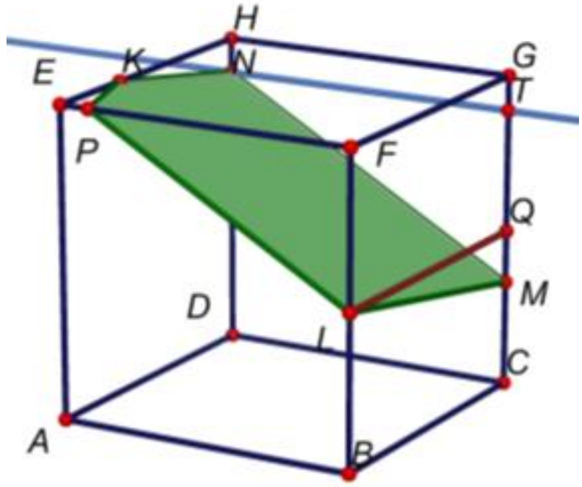
$$S_{KLMN} = \frac{1}{2}\overline{KL}\cdot\overline{ML} + \frac{1}{2}\overline{KN}\cdot\overline{MN}\cdot\sin\beta$$

$$S_{KLMN} = \frac{1}{2}\frac{\sqrt{3}}{4}\cdot\frac{\sqrt{7}}{4} + \frac{1}{2}\frac{3\sqrt{31}}{20}\cdot\frac{\sqrt{19}}{20}\cdot\frac{5\sqrt{651}}{589} = \frac{\sqrt{21}}{32} + \frac{3\sqrt{399}}{3040}$$

February 7-8: Let the cub ABCDEFGH of edge $\overline{AB} = 1$. Let K be the edge $\overline{EH}$ such that $\overline{HK} = 2\cdot\overline{EK}$. Let L the midpoint of the edge $\overline{BF}$. Let M be the edge $\overline{CG}$ so that $\overline{GM} = 2\cdot\overline{CM}$. Determine the sides of the section of the cube that determines the plane that passes through the points K, L, M.



Solution: Let Q be the midpoint of the edge $\overline{CG}$



$$\overline{QM} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\overline{LM} = \sqrt{\left(\frac{1}{6}\right)^2 + 1^2} = \frac{\sqrt{37}}{6}$$

The triangles $\overset{\Delta}{LQM}, \overset{\Delta}{KHN}$ are similar. Applying Thales' theorem:

$$\frac{\overline{HN}}{\frac{1}{6}} = \frac{1}{\frac{2}{3}}, \overline{HN} = \frac{1}{9}$$

$$\overline{KN} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{9}\right)^2} = \frac{\sqrt{37}}{9}$$

Let T be the projection of N on the edge $\overline{CG}$

$$\overline{MT} = \frac{2}{3} - \frac{1}{9} = \frac{5}{9}$$

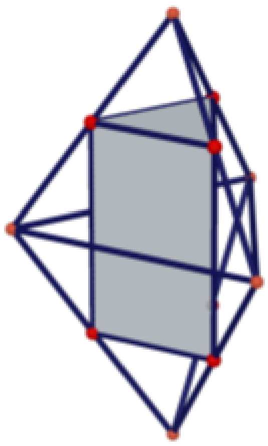
$$\overline{MN} = \sqrt{\left(\frac{5}{9}\right)^2 + 1^2} = \frac{\sqrt{106}}{9}$$

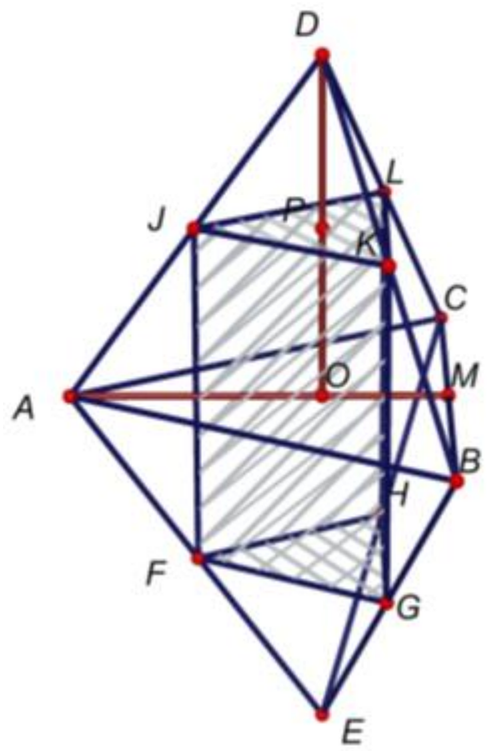
The triangles $\overset{\Delta}{NTM}, \overset{\Delta}{PEL}$ are similar. Applying Thales' theorem

$$\overline{PL} = \sqrt{\left(\frac{9}{10}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{106}}{10}$$

$$\overline{KP} = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{109}}{30}$$

February 9-16: Two regular tetrahedrons are joined by one face. Determine the ratio of the volumes of the vertex prism to the midpoints of the edges of the tetrahedra and the sum of the volumes of the two tetrahedral





Solution: Let ABCD, ABCE be regular tetrahedral with edge $\overline{AB} = a$. Let O be the centre of the common base. $\overset{\Delta}{ABC}$. Let $\overline{OD} = h$

Let be the regular prism FGHJKL of edge of the base $\overline{JK} = \frac{1}{2}\overline{AB} = \frac{1}{2}a$. The line OD intersects the base of the prism $\overset{\Delta}{JKL}$ in the point P. Let $\overline{OP} = x$. The height of the prism is 2x. Let M be the midpoint of the edge $\overline{BC}$

$$\overline{AM} = \frac{\sqrt{3}}{2}a$$

Applying the barycentre property:

$$\overline{AO} = \frac{2}{3}\overline{AM} = \frac{\sqrt{3}}{3}a$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{AOD}$

$$a^2 = \frac{1}{3}a^2 + \overline{OD}^2$$

$$\overline{OD} = \frac{\sqrt{6}}{3}a$$

$$\overline{OP} = x = \frac{1}{2}\overline{OD} = \frac{\sqrt{6}}{6}a$$

$$S_{ABC} = \frac{\sqrt{3}}{4}a^2, S_{KLM} = \frac{1}{4}S_{ABC} = \frac{\sqrt{3}}{16}a^2$$

The volume of the sum of the two tetrahedral is:

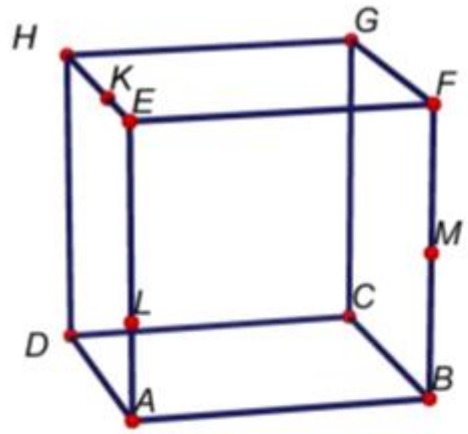
$$V_{ABCDE} = 2 \cdot \frac{1}{3}\frac{\sqrt{3}}{4}a^2 \cdot \frac{\sqrt{6}}{3}a = \frac{\sqrt{2}}{6}a^3$$

The volume of the prism is:

$$V_{FGHJKL} = \frac{\sqrt{3}}{16}a^2 \cdot 2\frac{\sqrt{6}}{6}a = \frac{\sqrt{2}}{16}a^3$$

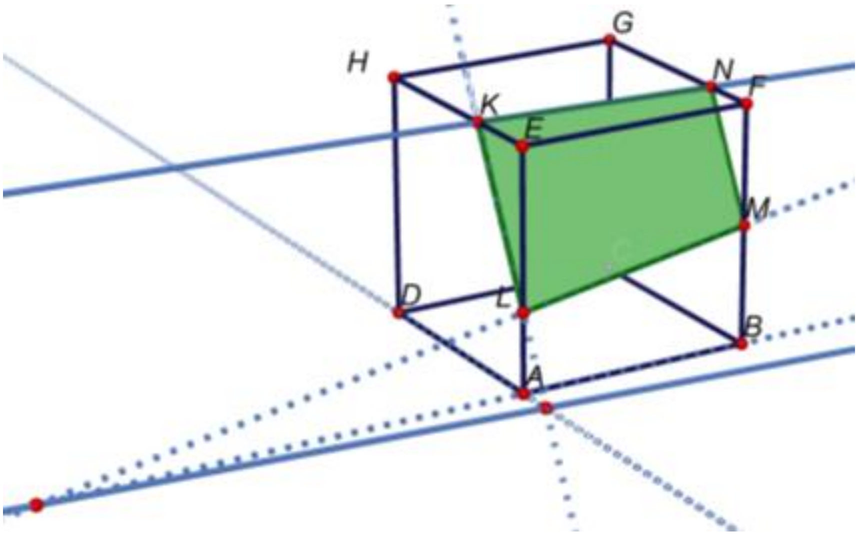
The volume ratio is:

$$\frac{V_{FGHJKL}}{V_{ABCDE}} = \frac{\frac{\sqrt{2}}{16}a^3}{\frac{\sqrt{2}}{6}a^3} = \frac{3}{8}$$



February 10-11: Let ABCDEFGH be a cube with edge $\overline{\mathrm{AB}} = 1$. Let K of the edge $\overline{\mathrm{EH}}$ so that $\overline{\mathrm{HK}} = 2 \cdot \overline{\mathrm{EK}}$. Let L of the edge $\overline{\mathrm{AE}}$so that $\overline{\mathrm{EL}} = 2 \cdot \overline{\mathrm{AL}}$. Let M be the midpoint of the edge $\overline{\mathrm{BF}}$. Determine the perimeter and the area of the section of the cube that determines the plane that passes through the points K, L, M.

Solution:



The section is the KLMN trapezoid where N belongs to the edge $\overline{\mathrm{FG}}$.

$$\overline{\mathrm{EK}} = \overline{\mathrm{AL}} = \frac{1}{3}, \overline{\mathrm{MF}} = \frac{1}{2}$$

The triangles $\overset{\Delta}{\mathrm{LEK}}, \overset{\Delta}{\mathrm{MFN}}$ they are similar. Applying Thales' theorem:

$$\frac{\overline{\mathrm{FN}}}{\frac{1}{3}} = \frac{\frac{1}{2}}{\frac{2}{3}}, \overline{\mathrm{FN}} = \frac{1}{4}$$

$$\overline{\mathrm{KL}} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

$$\overline{\mathrm{LM}} = \sqrt{\left(\frac{2}{3} - \frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{37}}{6}$$

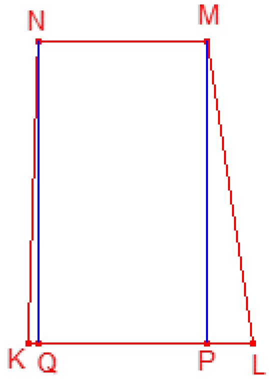
$$\overline{\mathrm{KN}} = \sqrt{\left(\frac{1}{3} - \frac{1}{4}\right)^2 + 1^2} = \frac{\sqrt{145}}{12}$$

$$\overline{\mathrm{MN}} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{\sqrt{5}}{4}$$

The perimeter of the KLMN trapezoid is:

$$P_{KLMN} = \frac{7\sqrt{5}}{3} + \frac{\sqrt{37}}{6} + \frac{\sqrt{145}}{12}$$

Let P be the projection of M onto $\overline{KL}$. Let Q be the projection of N onto $\overline{KL}$. Let $x = \overline{PL}$, then, $\overline{KQ} = \frac{\sqrt{5}}{12} - x$. Let $h = \overline{PM}$ the height of the KLMN trapeze. Applying the Pythagorean theorem to right triangles $\overset{\Delta}{MPL}$, $\overset{\Delta}{NQK}$



$$h^2 = \frac{37}{36} - x^2$$

$$h^2 = \frac{145}{144} - \left(\frac{\sqrt{5}}{12} - x\right)^2$$

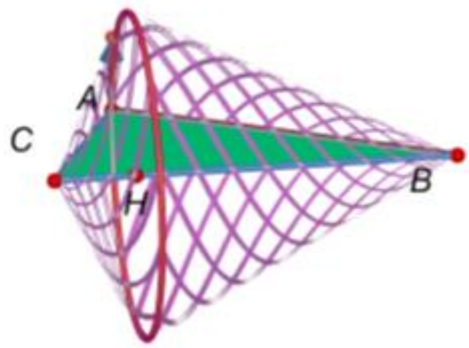
Solving the system formed by the two equations:

$$\begin{cases} x = \frac{\sqrt{5}}{15} \\ h = \frac{\sqrt{905}}{30} \end{cases}$$

The area of the KLMN trapezoid is:

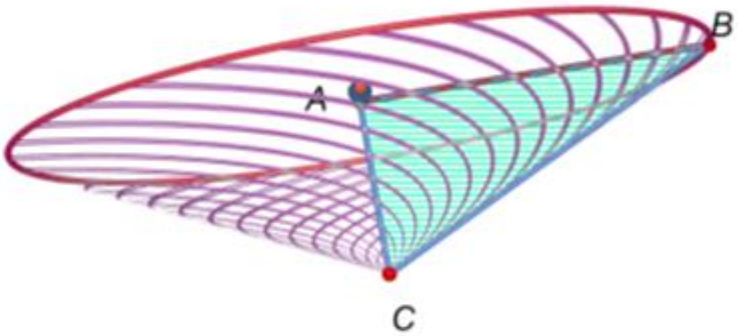
$$S_{KLMN} = \frac{\frac{\sqrt{5}}{3} + \frac{\sqrt{5}}{4}}{2} \frac{\sqrt{905}}{30} = \frac{7\sqrt{181}}{144}$$

February 14-15: The hypotenuse of a right triangle is 5. Determine the measure of the legs knowing that the volumes generated by the triangle when it revolves around the legs are one twice the other. Calculate the volume of the two cones. Determine the volume of the double cone generated by the triangle turning on the hypotenuse.



Solution: Be the right triangle $\overset{\Delta}{ABC}$ of hypotenuse $\overline{BC} = 5$. Let V_b the volume of the cone generated by rotating the triangle about the leg $b = \overline{AC}$. The radius is $c = \overline{AB}$. The volume is:

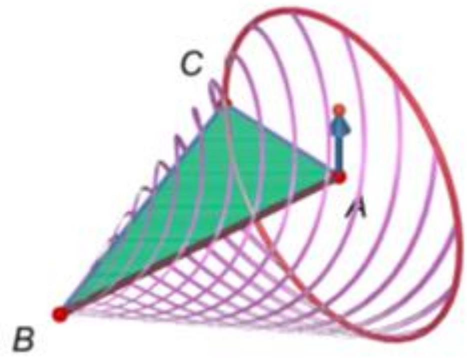
$$V_b = \frac{1}{3}\pi c^2 b$$



Let us suppose that the volume V_b is twice the volume V_c

$$\frac{1}{3}\pi c^2 b = 2 \cdot \frac{1}{3}\pi b^2 c$$

Simplified: $c = 2b$. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ABC}$:



$$b^2 + (2b)^2 = 5^2$$

Solving the equation:

$$b = \sqrt{5}, c = 2\sqrt{5}$$

The volumes of the cones are:

$$V_b = \frac{1}{3}\pi c^2 b = \frac{\pi}{3} 20\sqrt{5} = 46.83$$

$$V_c = \frac{1}{3}\pi b^2 c = \frac{\pi}{3} 10\sqrt{5} = 23.42$$

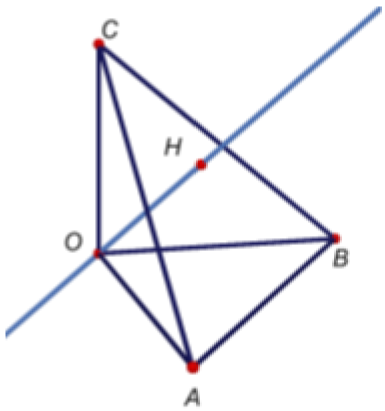
The radius of the two cones, which are generated by the rotation of the right triangle about the hypotenuse, is equal to the height of the triangle about the hypotenuse. Let $h = \overline{AH}$ the height above the hypotenuse. The area of the right triangle $\overset{\Delta}{ABC}$ is:

$$S_{ABC} = \frac{5h}{2} = \frac{\sqrt{5} \cdot 2\sqrt{5}}{2}$$

Solving the equation: $h = 2$. The volume of the double cone is:

$$V_a = \frac{1}{3}\pi h^2 \cdot 5 = \frac{20\pi}{3} \approx 20.94$$

February 17-24: The edges coming out of the vertex O of the tetrahedron OABC are perpendicular two by two.



Prove that the orthogonal projection H of O onto the face is the orthocentre of the triangle $\overset{\Delta}{ABC}$

Prove that $\frac{1}{\overline{OH}^2} = \frac{1}{\overline{OA}^2} + \frac{1}{\overline{OB}^2} + \frac{1}{\overline{OC}^2}$.

Prove that the symmetry of O with respect to the barycentre of the tetrahedron is the center of the sphere circumscribed to the tetrahedron

Solution: Consider the tetrahedron OABC with the following Cartesian coordinates:

$$O(0,0,0),\ A(a,0,0),\ B(0,b,0),\ C(0,0,c) \quad \overrightarrow{AB} = (-a,b,0),\ \overrightarrow{AC} = (-a,0,c).$$

The implicit equation of the plane containing points A, B, C is:

$$\Pi_{ABC} \equiv \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0.$$

Simplifying: $\Pi_{ABC} \equiv bcx + acy + abz - abc = 0$.

The normal vector is: $v = (bc, ac, ab)$.

The line OH has the equation: $r_{OH} \equiv (x, y, z) = \alpha(bc, ac, ab)$.

Solving the system formed by the line and the plane, the coordinates of point H are determined. Its coordinates are:

$$H\left(\frac{ab^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{a^2bc^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{a^2b^2c}{a^2b^2+b^2c^2+c^2a^2}\right).$$

$$\overrightarrow{CH} = \frac{1}{a^2b^2+b^2c^2+c^2a^2}\left(ab^2c^2, a^2bc^2, -b^2c^3-a^2c^3\right).$$

Let's see what $\overrightarrow{CH}$ is orthogonal to $\overrightarrow{AB} = (-a, b, 0)$. Let's calculate its scalar product:

$$\overrightarrow{CH}\cdot\overrightarrow{AB} = \frac{1}{a^2b^2+b^2c^2+c^2a^2}\left(ab^2c^2, a^2bc^2, -b^2c^3-a^2c^3\right)(-a, b, 0) = 0.$$

Analogously, $\overrightarrow{AH}$ is orthogonal to $\overrightarrow{BC}$. So H is the orthocentre of the triangle $\overset{\Delta}{ABC}$.

Let's go for the second statement:

$$\overrightarrow{OH} = \frac{1}{a^2b^2+b^2c^2+c^2a^2}\left(ab^2c^2, a^2bc^2, a^2b^2c\right).$$

$$\left\|\overrightarrow{OH}\right\| = \frac{1}{a^2b^2+b^2c^2+c^2a^2}\left\|\left(ab^2c^2, a^2bc^2, a^2b^2c\right)\right\| = \frac{abc}{a^2b^2+b^2c^2+c^2a^2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$\frac{1}{\left\|\overrightarrow{OH}\right\|} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}. \quad \frac{1}{\left\|\overrightarrow{OH}\right\|^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}. \quad \frac{1}{\left\|\overrightarrow{OA}\right\|^2} = \frac{1}{a^2}, \ \frac{1}{\left\|\overrightarrow{OB}\right\|^2} = \frac{1}{b^2}, \ \frac{1}{\left\|\overrightarrow{OC}\right\|^2} = \frac{1}{c^2}.$$

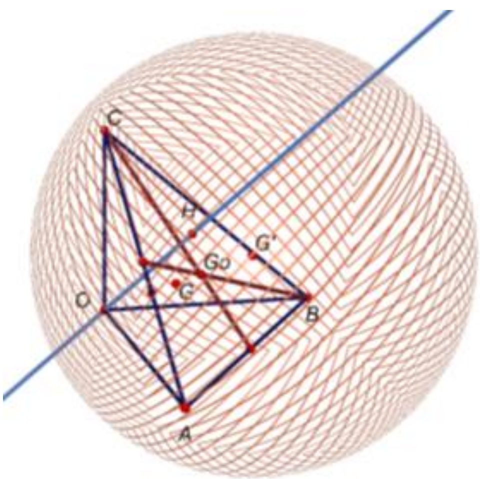
So, $\dfrac{1}{\overline{OH}^2} = \dfrac{1}{\overline{OA}^2}+\dfrac{1}{\overline{OB}^2}+\dfrac{1}{\overline{OC}^2}$.

Let's go to the third statement. the center of gravity G_O of the face $\overset{\Delta}{ABC}$ has coordinates:

$$G_O\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right).$$

The barycentre G of the tetrahedron belongs to the segment $\overline{OG_O}$ so that $\overrightarrow{OG} = \frac{3}{4}\overrightarrow{OG_O}$. Let $G(x, y, z)$.

$(x, y, z) = \dfrac{3}{4}\left(\dfrac{a}{3}, \dfrac{b}{3}, \dfrac{c}{3}\right)$. The coordinates of the barycentre of the tetrahedron are:



$$G\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right).$$

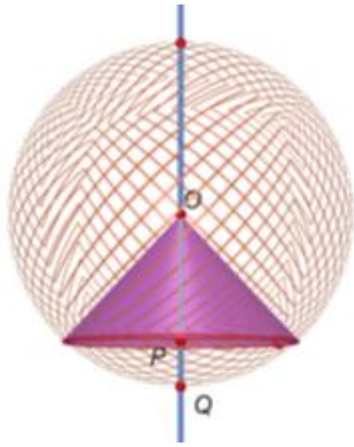
Let $G'(x, y, z)$ the symmetric of O with respect to G.

$$\overrightarrow{OG'} = 2\cdot\overrightarrow{OG}.$$

$(x, y, z) = 2\left(\dfrac{a}{4}, \dfrac{b}{4}, \dfrac{c}{4}\right)$. The coordinates of G' are:

$$G'\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right).$$

Let us note that $\overline{G'O} = \overline{G'A} = \overline{G'B} = \overline{G'C} = \frac{1}{2}\sqrt{a^2+b^2+c^2}$, then G' is the center of the sphere circumscribed to the tetrahedron OABC



February 18-19: A sphere of radius r has a cone inscribed with its vertex at the center of the sphere and angle 2α at its vertex.

Determine the area and volume of the area of the sphere that intersects the cone.

Problem proposed by Joan Galiana, student and mathematician

Solution: For the area, we have: The cone cuts the sphere into two spherical caps. We will consider the smallest. (The largest would be easy to calculate from the difference between the surface of the sphere and the smallest cap). Let be the axis of symmetry of the cone that cuts the sphere at point Q and passes through the center of the base of the cone P. Let $h = \overline{PQ}$ the height of the cap. The area of the cap is:

$$S_{casquet} = 2\pi r \cdot h\,.\quad \overline{OP} = r \cdot \cos\alpha\,.\quad \overline{PQ} = (1-\cos\alpha)\,r\,.$$

The area of the cap is:

$$S_{casquet} = 2\pi r \cdot h = 2\pi(1-\cos\alpha)r^2\,.$$

For the volume, we have: The cone determines two parts, a cap and the cone itself, and the other part, which is what remains of the previous one. The volume of the cap is:

$$V_{casquet} = \pi h^2\left(r - \frac{h}{3}\right).$$

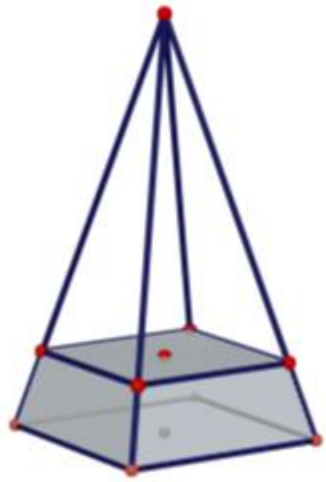
$$V_{casquet} = \pi h^2\left(r - \frac{h}{3}\right) = \frac{\pi}{3}(1-\cos\alpha)^2(2+\cos\alpha)r^3\,.$$

The radius of the cone is: $s = r \cdot \sin\alpha$. The height of the cone is $t = \overline{OP} = r \cdot \cos\alpha$. The volume of the cone is:

$$V_{casquet} = \frac{1}{3}\pi s^2 t = \frac{1}{3}\pi\cos^2\alpha \cdot \sin\alpha \cdot r^3\,.$$

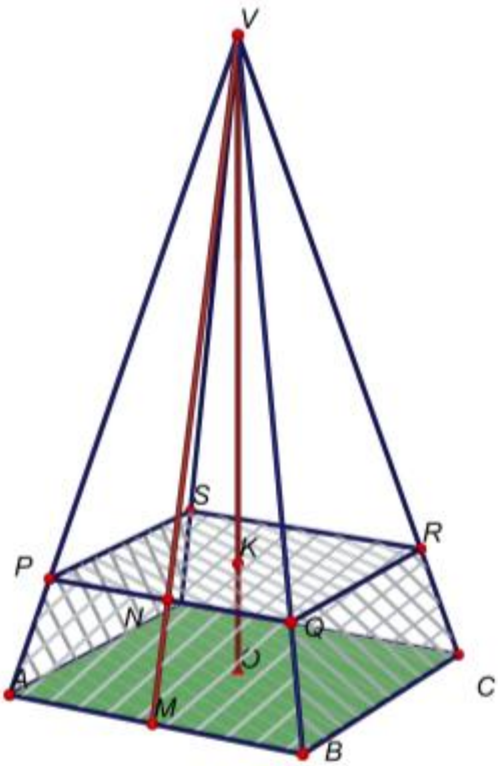
The volume of the bottom is:

$$V_{total} = V_{casquet} + V_{con} = \frac{\pi}{3}\left((1-\cos\alpha)^2(2+\cos\alpha)r^3 + \cos^2\alpha \cdot \sin\alpha\right)r^3$$



February 21-28: The height of a lateral face of a regular quadrangular pyramid is twice the edge of the base.

What percentage of this height of the pyramid (counting from the base) do we need to cut with a plane parallel to the base so that the total area of the lateral surface plus the top square of the resulting truncated pyramid equals half the lateral surface of the original pyramid.



Solution: Let $\overline{AB} = a$ the edge of the base of the pyramid ABCDV. Let $\overline{MV} = 2a$ the height of a lateral face. Let the frustum of the pyramid be ABCDPQRS. Let $\overline{PQ} = b$ the edge of the upper face of the trunk. Let O be the center of the base ABCD. Let K be the center of the basis PQRS. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{VMO}$. The height of the pyramid is:

$$H = \overline{VM} = \frac{\sqrt{15}}{2}a.$$

The lateral area of the pyramid is:

$$S_L = 4 \cdot \frac{1}{2}a \cdot 2a = 4a^2$$

Let N be the midpoint of the edge $\overline{PQ}$. The pyramids ABCDV, PQRSV are similar. Then:

$$\overline{NV} = 2b$$

The height of trapezoid ABQP is $\overline{MN}$.

$$\overline{MN} = 2(a - b)$$

The area of the trapezoid is:

$$S_{ABQP} = \frac{a+b}{2} \cdot 2(a - b)$$

The area of the lateral surface plus the upper square of the frustum of the pyramid is: $S_2 = 4(a+b)(a-b) + b^2$, the total area of the lateral surface plus the upper square of the resulting frustum of the pyramid

$$S_2 = \frac{1}{2}S_L$$

$$4(a+b)(a-b) + b^2 = 2a^2$$

simplifying:

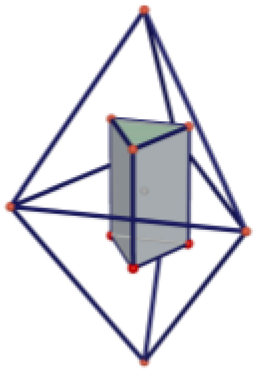
$$\frac{b}{a}=\sqrt{\frac{2}{3}}$$

Let $h=\overline{OK}$ the height of the frustum of the pyramid. The pyramids ABCDV, PQRSV are similar. Then,

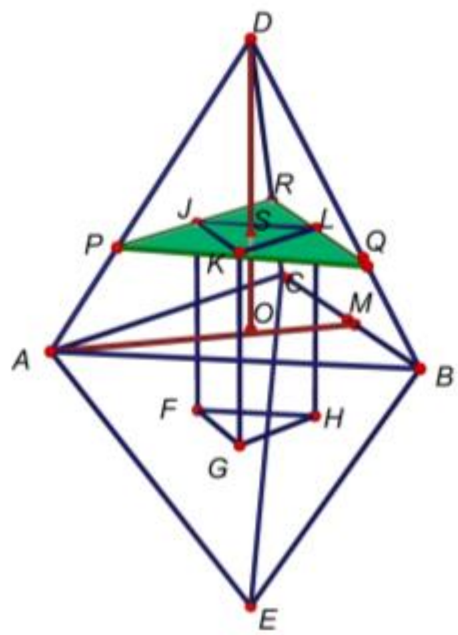
$$\frac{b}{a}=\frac{H-h}{H}$$

$$\frac{h}{H}=1-\frac{b}{a}=1-\sqrt{\frac{b}{a}}=1-\sqrt{\frac{2}{3}}\approx 0.1835$$

That is, it cuts the initial pyramid at 18.35% of its height.



February 22-23: Given the regular double tetrahedron, determine the ratio between the volumes of the dual polyhedron (vertex prism the centers of the 6 faces) and the regular double tetrahedron.



Solution: Let the regular tetrahedra ABCD, ABCE be of edge $\overline{AB}=a$. Let O be the center of the common base $\overset{\Delta}{ABC}$. Let $\overline{OD}=h$. Let FGHJKL be the regular prism such that the vertices are the faces' centroid. The line OD intersects the base of the prism $\overset{\Delta}{JKL}$ at point S. Let $\overline{OS}=x$. The height of the prism is 2x. The plane containing the base $\overset{\Delta}{JKL}$ intersects the edges of tetrahedron ABCD at points P, Q, R. Let M be the midpoint of the edge $\overline{BC}$

$$\overline{AM}=\frac{\sqrt{3}}{2}a$$

Applying the barycentre property:

$$\overline{AO}=\frac{2}{3}\overline{AM}=\frac{\sqrt{3}}{3}a$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{AOD}$

$$a^2=\frac{1}{3}a^2+\overline{OD}^2$$

$$\overline{OD}=\frac{\sqrt{6}}{3}a$$

The volume of the sum of the two tetrahedra is:

$$V_{ABCDE}=2\cdot\frac{1}{3}\frac{\sqrt{3}}{4}a^2\cdot\frac{\sqrt{6}}{3}a=\frac{\sqrt{2}}{6}a^3$$

The equilateral triangles $\overset{\Delta}{ABD},\overset{\Delta}{PQD}$ are similar and of ratio 3:2. Then:

$$\overline{PQ} = \frac{2}{3}a$$

$$S_{JKL} = \frac{1}{4}S_{PQR} = \frac{1}{4}\frac{\sqrt{3}}{4}\left(\frac{2}{3}a\right)^2 = \frac{\sqrt{3}}{36}a^2$$

The right triangles $\overset{\Delta}{AOD}, \overset{\Delta}{PSD}$ are similar and have a ratio of 3:2. Then:

$$\overline{OS} = \overline{OD} - \overline{SD} = \frac{1}{3}\overline{OD} = \frac{1}{3}\frac{\sqrt{6}}{3}a = \frac{\sqrt{6}}{9}a$$

The volume of the prism is:

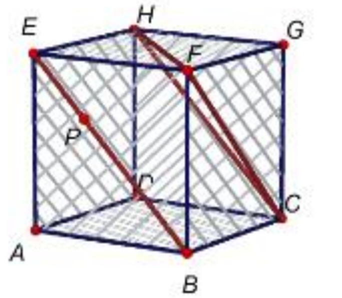
$$V_{FGHJKL} = \frac{\sqrt{3}}{36}a^2 \cdot 2\frac{\sqrt{6}}{9}a = \frac{\sqrt{2}}{54}a^3$$

The volume ratio is:

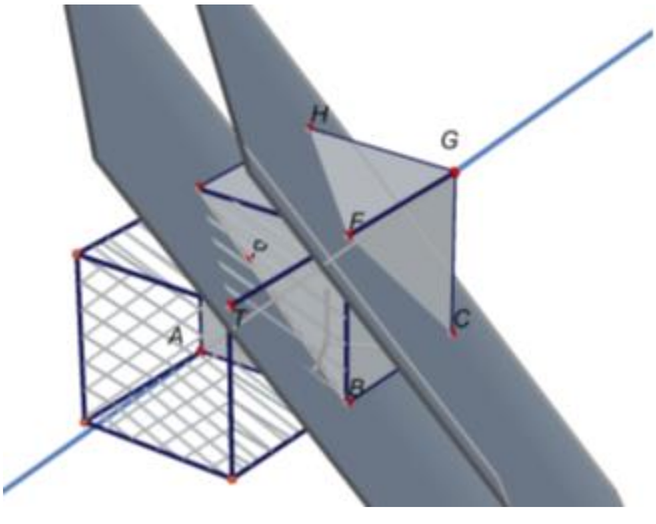
$$\frac{V_{FGHJKL}}{V_{ABCDE}} = \frac{\frac{\sqrt{2}}{54}a^3}{\frac{\sqrt{2}}{6}a^3} = \frac{1}{9}$$

February 25-26: Let ABCDEFGH be a cube of edge 1. Let P be a point on the segment $\overline{BE}$ such that $\overline{EP} : \overline{BE} = 1 : 3$.

Calculate the distance from point P to the plane determined by the vertices C, F, H of the cube.



Solution: The distance from the point P to the plane that determines the points C, F, H is equal to the distance of the plane that contains the line BE and is parallel to the previous one and the initial plane. That is, the plane that contains the points B, E, T (where T is the vertex of a cube attached to the previous one, see figure). We note that the diagonal AG is perpendicular to the two planes. Let h be the height of the tetrahedron CFHG above the base CFG.



The volume of the CFHG tetrahedron is:

$$V_{Tetraedre} = \frac{1}{3}\frac{1}{2}\overline{CG}^2 = \frac{1}{3}S_{CFH} \cdot h.$$

$$\frac{1}{3}\frac{1}{2} = \frac{1}{3}\frac{\sqrt{3}}{4}\left(\sqrt{2}\right)^2 \cdot h.$$

Solving the equation:

$$h = \frac{\sqrt{3}}{3}.$$

The distance between the two planes is:

$$d = \overline{AG} - 2h = \sqrt{3} - 2\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{3}.$$

Solution 2: Let be the cube with the following coordinates:

$$A(0,0,0),\ B(1,0,0),\ C(1,1,0),\ D(0,1,0).\ E(0,0,1),\ F(1,0,1),\ G(1,1,1),\ H(0,1,1).$$

The coordinates of P are:

$$P\left(\frac{1}{3}, 0, \frac{2}{3}\right).$$

$$n\,\overrightarrow{FH} = (-1,1,0),\ \overrightarrow{FC} = (0,1,-1).$$

The general equation of the plane passing through the points C, F, H is:

$$\Pi \equiv \begin{vmatrix} x-1 & y & z-1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0.$$

Simplifying: $\Pi \equiv x + y + x - 2 = 0$.

The distance from P to the plane is:

$$d(P,\Pi) \equiv \left| \frac{\frac{1}{3} + 0 + \frac{2}{3} - 2}{\sqrt{1^2 + 1^2 + 1^2}} \right| = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$