SOLUTIONS MAY 2022

PROBLEMS FOR THE PREPARATION OF THE OLYMPICS OF 3º AND 4º OF E.S.O. ORGANIZED BY THE FESPM IN 2004. ORGANISATION: JOSÉ COLÓN LACALLE. Retired teacher.

May 2-3: The sisters Laia and Aitana write their ages, one after the other, and get a four-digit number that is the square of their father's age. Nine years later they rewrite their ages in the same way and it happens again to be the square of their father's age. What is the age of Laia, Aitana and her father?

Solution: Let AB be the age of the first sister and CD the age of the second sister. If P is the age of the father, we will have, according to the statement:

$$\overline{\mathrm{ABCD}} = \overline{\mathrm{AB}} \cdot 100 + \overline{\mathrm{CD}} = \mathrm{P}^2 \quad (1)$$

And, nine years later, we will have:

$$(\overline{\mathrm{AB}} + 9) \cdot 100 + (\overline{\mathrm{CD}} + 9) = (\mathrm{P} + 9)^2$$

Expanding this last equation, we have:

$$\overline{\mathrm{AB}} \cdot 100 + 900 + \overline{\mathrm{CD}} + 9 = \mathrm{P}^2 + 18\mathrm{P} + 81$$

and taking into account equation (1), we arrive at:

$$909 = 18\mathrm{P} + 81 \;\; \Rightarrow \;\; \mathrm{P} = \frac{909 - 81}{18} = 46$$

That is, the father is 46 years old. Lastly, as:

$$46^2 = 2116$$

Laia's age is 21 years old and Aitana's is 16 years old

May 4: We have two hourglasses, one of 7 minutes and another of 11 minutes. What is the fastest method to control the cooking of a stew that should last 15 minutes?

Solution: We put to work simultaneously, the two clocks. When the 7-minute time is up, we turn on the fire. When the 11-minute clock ends, (11 – 7 =) 4 minutes of cooking will have elapsed. We let the 11-minute clock run again. When it ends, (4 + 11 =) 15 minutes of cooking will have elapsed.

May 5-6: The town of Benirredrà has a very strange set of speed limits. One kilometre from the town centre there is a sign that says 120 km/h, ½ kilometre from the centre, a sign says 60 km/h, 1/3 km from the centre there is a sign that announces 40 km/h, ¼ of a km from the centre there is a sign that says 30 km/h, at 1/5 of the centre a notice of 24 km/h and at a distance of 1/6 of a km the sign says 20 km/h. If you always travel at the speed limit, how long will it take to get to the centre of town from the first announcement?

Solution: We will have:

First stretch:

$$\begin{Bmatrix} \mathrm{e}_1 = 1 - \frac{1}{2} = \frac{1}{2}\mathrm{km} \\ \mathrm{v}_1 = 120\frac{\mathrm{km}}{\mathrm{h}} \end{Bmatrix} \Rightarrow \; \mathrm{t}_1 = \frac{\mathrm{e}_1}{\mathrm{v}_1} = \frac{\frac{1}{2}}{120}\mathrm{h} = \frac{60 \cdot 60}{2 \cdot 120}\mathrm{sg} = 15\ \mathrm{sg}$$

Second leg:

$$\left\{\begin{aligned} e_2 &= \frac{1}{2}-\frac{1}{3}=\frac{1}{6}\text{km} \\ v_2 &= 60\frac{\text{km}}{\text{h}} \end{aligned}\right\} \Rightarrow\ t_2=\frac{e_2}{v_2}=\frac{\frac{1}{6}}{60}\text{h}=\frac{60\cdot 60}{6\cdot 60}\text{sg}=10\text{ sg}$$

Third leg:

$$\left\{\begin{aligned} e_3 &= \frac{1}{3}-\frac{1}{4}=\frac{1}{12}\text{km} \\ v_3 &= 40\frac{\text{km}}{\text{h}} \end{aligned}\right\} \Rightarrow\ t_3=\frac{e_3}{v_3}=\frac{\frac{1}{12}}{40}\text{h}=\frac{60\cdot 60}{12\cdot 40}\text{sg}=7{,}5\text{ sg}$$

Fourth leg:

$$\left\{\begin{aligned} e_4 &= \frac{1}{4}-\frac{1}{5}=\frac{1}{20}\text{km} \\ v_4 &= 30\frac{\text{km}}{\text{h}} \end{aligned}\right\} \Rightarrow\ t_4=\frac{e_4}{v_4}=\frac{\frac{1}{20}}{30}\text{h}=\frac{60\cdot 60}{20\cdot 30}\text{sg}=6\text{ sg}$$

Fifth tranche:

$$\left\{\begin{aligned} e_5 &= \frac{1}{5}-\frac{1}{6}=\frac{1}{30}\text{km} \\ v_5 &= 24\frac{\text{km}}{\text{h}} \end{aligned}\right\} \Rightarrow\ t_5=\frac{e_5}{v_5}=\frac{\frac{1}{30}}{24}\text{h}=\frac{60\cdot 60}{30\cdot 24}\text{sg}=5\text{ sg}$$

Sixth tranche:

$$\left\{\begin{aligned} e_6 &= \frac{1}{6}\text{km} \\ v_5 &= 20\frac{\text{km}}{\text{h}} \end{aligned}\right\} \Rightarrow\ t_6=\frac{e_6}{v_6}=\frac{\frac{1}{6}}{20}\text{h}=\frac{60\cdot 60}{6\cdot 20}\text{sg}=30\text{ sg}$$

Total:

$$t_1+t_2+t_3+t_4+t_5+t_6=15+10+7{,}5+6+5+30=73{,}5\text{ sg}=1\text{m }13{,}5\text{ sg}$$

May 7: Find five consecutive integers such that the sum of squares of the first three is equal to the sum of squares of the last two

Solution: Let:

$$x-2, x-1, x, x+1, x+2$$

consecutive integers. From the statement we have that it must be fulfilled that:

$$(x-2)^2+(x-1)^2+x^2=(x+1)^2+(x+2)^2$$

Which leads to:

$$0=12x-x^2=(12-x)\cdot x\ \Rightarrow\ \begin{cases} x=0 \\ x=12 \end{cases}$$

The integers in the statement are: -2, -1, 0, 1 y 2, o 10, 11, 12, 13 y 14

May 9: A month ago, 10% of a population had a disease and 90% did not. After a month, 10% of sick people are cured and 10% of people who did not have it became sick. What % of the population does not have the disease?

Solution: Suppose the total population is t people. We can generate the following contingency table:

		Nowadays		
		E	$\overline{E}$	
1 month ago	E		**0,01·t**	**0,1·t**
	$\overline{E}$	**0,09·t**		**0,9·t**

We have from the statement:

Total first row: A month ago 10% of a population had a disease:

$$10\% \text{ de } t = 0{,}1 \cdot t$$

Total second row: A month ago 90% of a population did not have the disease:

$$90\% \text{ de } t = 0{,}9 \cdot t$$

Cell first row, second column: After a month (currently), 10% of sick people are cured:

$$10\% \text{ de } 0{,}1 \cdot t = 0{,}01 \cdot t$$

Cell second row, first column: After the month (currently), 10% of the people who did not have it became ill:

$$10\% \text{ de } 0{,}9 \cdot t = 0{,}09 \cdot t$$

		Nowadays		
		E	$\overline{E}$	
1 month ago	E	**0,09·t**	**0,01·t**	**0,1·t**
	$\overline{E}$	**0,09·t**	**0,81·t**	**0,9·t**
		0,18·t	**0,82·t**	**t**

And finally, we complete the contingency table:

Then, currently there are 18% of patients and 82% of the population does not have the disease

May 10-11: A horse and a mule walked together, carrying heavy sacks on their backs. The nag lamented his annoying burden, to which the mule said: What are you complaining about? If I took a sack from you, my burden would be double yours. Instead, if I give you a sack, your load would be equal to mine. How many bags does each one carry?

Solution: A typical word problem: a situation is described and a question is asked about it: Let x be the number of bags carried by the horse and y be the number of bags carried by the mule. We have:

"If I took a sack from you, my burden would be double yours."

$$y + 1 = 2(x - 1)$$

“If I give you a sack, your load would be equal to mine”

$$y - 1 = x + 1$$

The system is generated:

$$\begin{cases} y + 1 = 2(x - 1) \\ y - 1 = x + 1 \end{cases}$$

Solving for y in the second equation and substituting in the first we arrive at: x = 5 and by substituting in the second we arrive at y = 7.

The mule carries 7 sacks and the horse carries 5 sacks.

May 12: An NGO sells 140 tickets to raise funds. Some tickets were sold at the original price (one integer), but the others were sold at half the original price. €2,001 was raised. What is the starting price of the tickets?

Solution: From the statement we have:

$$140 \text{ tickets} \begin{cases} x\ (\leq 140) \text{ to the price } p \in \mathbb{N} \\ 140 - x \text{ to the price } \frac{p}{2} \end{cases}$$

Since the winnings are €2001, we will have:

$$xp + (140 - x)\frac{p}{2} = 2001 \;\Rightarrow\; xp + 140p = 4002 \;\Rightarrow\; p = \frac{4002}{x + 140}$$

And since $p \in \mathbb{N}$, x + 140 is a positive divisor of 4002. Since $0 \leq x \leq 140 \Rightarrow 140 \leq x+140 \leq 2\cdot 140 = 280$. Since 4002 =2 3 23 29, the divisors of 4002 are: 1, 2, 3, 6, 23, 29, 46, 58, 69, 87, 138, 174, 1334, ····. The only divisor between 140 and 280 is 174. From where x = 174 – 140 = 34 and

$$p = \frac{4002}{174} = 23$$

That is, 34 tickets were sold at 23 euros and the rest (140 – 34 =) 106 tickets at half price (23/2 =) 11.5 €.

May 13-14: In each station of a railway network, as many different tickets are sold as there are stations to which you can go or from which you can come (the outward and return tickets are different). A new line with several stations is inaugurated and this forces the printing of 34 new tickets. How many stations were there and how many have been inaugurated?

Solution: Suppose that before the inauguration of the new line there were n stations in operation. The number of tickets to be printed was n·(n – 1) (because round-trip tickets are different). If with the incorporation of the new line x stations are added, the number of different tickets becomes (n + x)·(n + x – 1). Subtracting, we will have:

$$(n + x)\cdot(n + x - 1) - n\cdot(n - 1) = 34$$

which, developing, becomes:

$$(2n + x - 1)\cdot x = 1\cdot 2\cdot 17 \;\Rightarrow\; \begin{cases} x = 1 \Rightarrow 2n + x - 1 = 34 \Rightarrow 2n = 34 \Rightarrow n = 17 \\ x = 2 \Rightarrow 2n + x - 1 = 17 \Rightarrow 2n = 16 \Rightarrow n = 8 \\ x = 17 \Rightarrow 2n + x - 1 = 2 \Rightarrow 2n = -14 \Rightarrow \text{NO} \\ x = 34 \Rightarrow 2n + x - 1 = 1 \Rightarrow 2n = -34 \Rightarrow \text{NO} \end{cases}$$

Since the statement speaks of several new stations, it must be x = 2 and n = 8. That is, there were 8 stations and two new stations have been added.

May 16-17: Two players A and B take turns playing the following game: You have a pile of 2021 stones. On his first turn A chooses a divisor of 2021 and removes that number of stones from the pile. B then chooses a divisor of the number of stones remaining and removes that number of stones from the pile, and so on. The player who removes the last stone loses. Show that one of the players has a winning strategy and describe that strategy.

Solution: B wins if A hands him an even number of stones and B subtracts the product of (all or some) of the odd ones from the prime factorial decomposition of the number A gives him and gives A an odd number (If in the number he receives B there are no odd factors, B subtracts 1, that is, chooses divisor 1).

Indeed, A always gives an even number to B and A always receives an odd number, since:

1.-

$$43 \cdot 47 = 2021 \begin{cases} \xrightarrow[A(-1)]{} 2020 = 2^2 \cdot 5 \cdot 101 \quad \text{(even number)} \\ \xrightarrow[A(-2021)]{} 0 \qquad\qquad\qquad \text{(B win)} \\ \xrightarrow[A(-43)]{} 1978 = 2 \cdot 23 \cdot 43 \quad \text{(even number)} \\ \xrightarrow[A(-47)]{} 1974 = 2 \cdot 3 \cdot 7 \cdot 47 \quad \text{(even number)} \end{cases}$$

2.- If B receives $2^{\alpha}\cdot$I and subtract I' (being I and I' product of odd) give A the number:

$$2^{\alpha}I'I'' - I' = (2^{\alpha}I'' - 1) \cdot I' \quad ((\text{even} - 1) \cdot \text{odd} = \text{odd})$$

3.- If B receives 2^{δ}, B subtracts 1 and makes A odd (2^{δ} - 1 = even – odd = odd)

4.- Since the number given by one or another player is subtracted from another number, the process is descending.

5.- There will come a time when B transmits to A an odd with only two divisors: 1 and 3

$$\begin{cases} \xrightarrow[A(-1)]{} 2 \xrightarrow[B(-1)]{} 1 \xrightarrow[A(-1)]{} 0 \quad \text{B wins} \\ \xrightarrow[A(-3)]{} 0 \qquad\qquad\qquad \text{B wins} \end{cases}$$

May 18: In cubeland, the planets are cubes. Aitana has one with an edge of 1 km. If its atmosphere is 500 m high at all points, what is the volume of the atmosphere?

Solution: The planet's atmosphere is made up of:

A. 6 parallelepipeds (one for each of the faces of the cube that forms the planet) with dimensions 1 km x 1km x 0.5 km. The total volume of these parallelepipeds is (6·0.5 km3 =) 3 km^3.
B. 12 quarter cylinders (a quarter for each of the edges of the cube), that is, three cylinders with dimensions 1 km in height and 0.5 km in radius. The total volume of these cylinders is (3·π·0.52·1 =) 0.75·π km^3.
C. Eight eighths of a sphere (one eighth for each vertex of the cube), that is, a sphere with total volume:

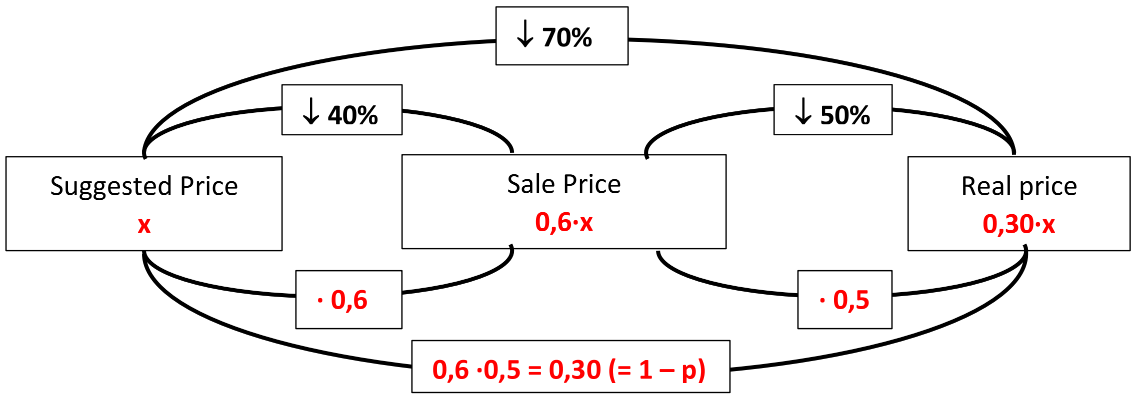
$$\left(\frac{4}{3} \cdot \pi \cdot 0{,}5^3 =\right) \frac{\pi}{6} \ \text{km}^3$$

The total volume of the atmosphere is:

$$3 + 0{,}75\pi + \frac{\pi}{6} = \frac{18 + 5{,}5\pi}{6} \approx 5{,}8797 \ldots. \ \text{km}^3$$

May 19-20: The selling price of a coat was 40% less than the manufacturer's suggested price. Laia bought the coat for half the sale price. In what percentage is the value that Laia paid for the coat lower than the price suggested by the manufacturer?

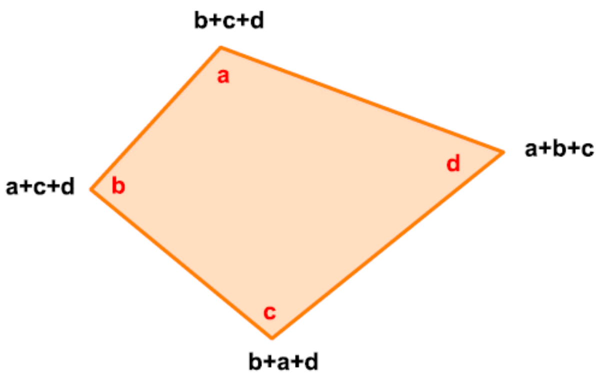
Solution:



$$0{,}30 = 1 - p \quad \Rightarrow \quad p = 0{,}7 = 70\%$$

May 21-28: At the vertices of a quadrilateral a secret number is written with invisible ink and with visible ink the sum of the invisible numbers of the other three vertices. Can you give a rule to calculate the invisible numbers from the visible numbers?

Solution: Let us imagine the attached figure where, in each vertex is, in black, the sum of the three invisible numbers that are in the other three vertices and in red are the four invisible numbers. Let's see how to locate the value of the invisible number a:



1.- We add the visible numbers of the other vertices, obtaining:

$$3a + 2(b + c + d)$$

2.- We subtract twice the visible number of the vertex in question and obtain:

$$3a + 2(b + c + d) - 2(b + c + d) = 3a$$

3.- Finally, we divide by 3 and get:

$$\frac{3a}{3} = a$$

That is, to find the invisible number of a vertex:

We have to add the visible numbers of the other vertices. Subtract twice the visible number of the vertex from the number obtained. And finally divide by 3.

Note: The procedure described here can be generalized to polygons of any number of vertices.

May 23: Find, if possible, the largest and smallest natural whose sum of digits is 2022

Solution: Let us first find the natural minor with sum of digits 2022. Since

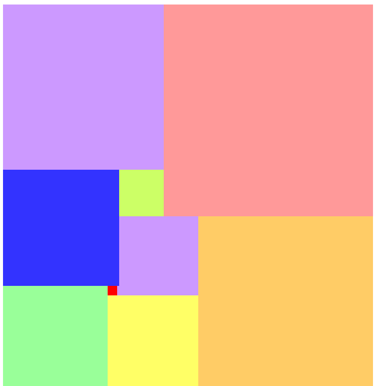
2022	9
6	224

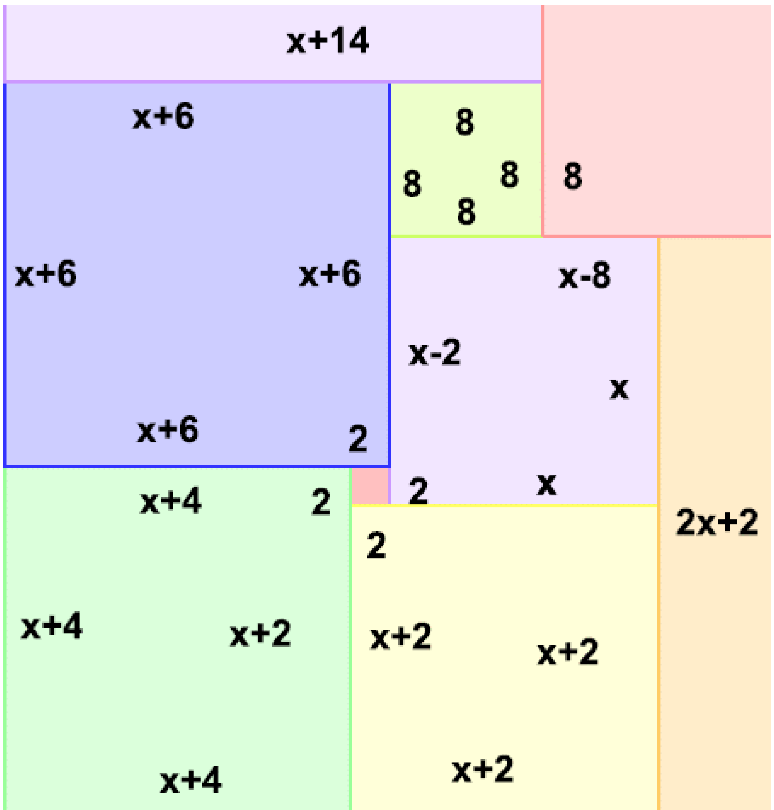
The number formed by 224 nines preceded by a six fulfills that the sum of its digits is 2022 and obviously it is the smallest that fulfills this condition.:

$$6\underbrace{99\ldots99}_{224}$$

There is no larger natural number with the sum of figures 2022, since assuming that it exists, it is enough to multiply it by 10 (or by 100 or by 1000, …..) to obtain a larger number with the same sum of figures.

<u>May 24-25:</u> The rectangle in the figure is divided into nine squares. Calculate its height and length knowing that the smallest square has side 2 cm.





<u>Solution:</u> The sides of the various squares are going to be obtained starting from the side of the only known square, the red one, whose side measures 2.

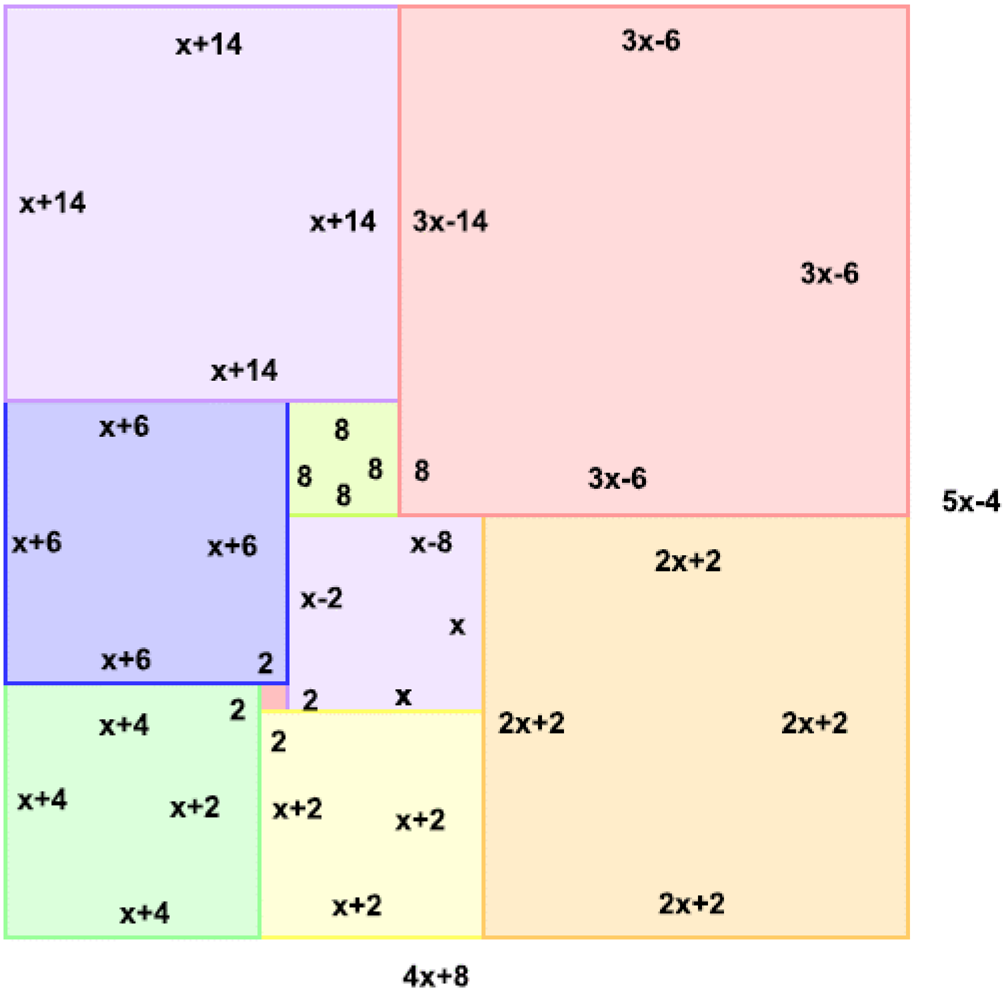
Let x be the side of the square to the right of side 2. Then the side of the square below will be x + 2. The side of the square to its left will be x + 4. The side of the square above will be x + 6. So the green square to its right will be:

$$x + 6 - (x - 2) = 8$$

So the side of the purple square above the last two will be:

$$x + 6 + 8 = x + 14$$

We go down. The side of the square to the right of the squares with sides x and x + 2 will be: 2x + 2



The side of the square above it will be:

$$2x + 2 + x - 8 = 3x - 6$$

This square shares with the square to its left a side of:

$$3x - 6 - 8 = 3x - 14$$

So:

$$x + 14 = 3x - 14 \;\; \Rightarrow \;\; x = 14$$

The dimensions of the large rectangle are:

$$\begin{cases} 2x + 2 + 3x - 6 = 5x - 4 = 5 \cdot 14 - 4 = 66 \\ x + 4 + x + 2 + 2x + 2 = 4x + 8 = 4 \cdot 14 + 8 = 64 \end{cases}$$

__May 26-27:__ A person has €500 in a checking account at a bank. He can make two movements indefinitely, as long as he has money in the account: withdraw €300 or deposit €198. What is the maximum amount of money you can withdraw from your account?

__Solution:__ Apparently it seems that the most money we can withdraw is €300, but if after the withdrawal we deposit €198 (there would be (200 + 198 =) €398 in the bank) we could make a withdrawal of €300 and thus we would have (300 – 198 + 300 =) €402 in cash, which seems to be the maximum. Continuing with this process we generate the following table:

	Cash	In bank	Action
a_0	0	500	Extract 300 €
a_1	300	200	deposit 198 €
a_2	102	398	Extract 300 €
$a_3 = b_1$	402	98	deposit 198 €
a_4	204	296	deposit 198 €
a_5	6	494	Extract 300 €
a_6	306	194	deposit 198 €
a_7	108	392	Extract 300 €
$a_8 = b_2$	408	92	deposit 198 €
a_9	210	290	deposit 198 €
a_{10}	12	488	Extract 300 €
a_{11}	312	188	deposit 198 €
a_{12}	114	386	Extract 300 €
$a_{13} = b_3$	414	86	deposit 198 €
a_{14}	216	284	deposit 198 €
a_{15}	18	482	Extract 300 €
a_{16}	318	182	deposit 198 €
a_{17}	120	380	Extract 300 €
$a_{18} = b_4$	420	80	.....
.....	.....	.....	....

We see that a "recurrence" occurs: every five bank movements there is a peak of cash:

$$b_1 = a_3 = 402;\ b_2 = a_8 = 408;\ b_3 = a_{13} = 414;\ b_4 = a_{18} = 420;\ \cdots;\ b_n = a_{5(n-1)+3}$$
$$= 402 + 6(n-1) = 6n + 396$$

It is an arithmetic progression of first term 402 and difference d = 6. Furthermore, necessarily, $b_n \leq 500$

$$6n + 396 \leq 500 \ \Rightarrow\ n \leq 17,\hat{3} \ \Rightarrow\ n = 17 \ \Rightarrow\ b_{17} = a_{83} = 6 \cdot 17 + 396 = 498\ €$$

€2 remaining in the bank (= 500 – 498). And then the whole process is repeated:

	Cash	In bank	Action
$a_{83} = b_{17}$	498	2	deposit 198 €
$a_{84} = a_1$	300	200	deposit 198 €
....	.....	.....	......

In short: The largest amount of money that can be obtained in cash is €498.

May 30-31: A basketball club has a men's section and a women's section. The arithmetic mean weight of the boys in the male section is 90 kilos, the arithmetic mean weight of the girls in the female section is 65 kilos. The arithmetic mean of the weight of all the components of the club is 75 kilos. Are there more girls than boys? What proportion of girls are there among all the players in the club?

Solution: Let:

$$\sum\nolimits_{H.} \text{the weight of men}$$
$$\sum\nolimits_{D.} \text{the weight of women}$$
$$n_H \text{The number of male players}$$
$$n_D \text{ the number of female players}$$

then:

$$90 = \frac{\sum_{H.}}{n_H}; \quad 65 = \frac{\sum_{D.}}{n_D}; \quad 75 = \frac{\sum_{H.} + \sum_{D.}}{n_H + n_D} = \frac{90n_H + 65n_D}{n_H + n_D} \quad \Rightarrow \quad 75n_H + 75n_D = 90n_H + 65n_D$$
$$\Rightarrow \quad 10n_D = 15n_H \quad \Rightarrow \quad n_D = 1{,}5n_H$$

That is, there are 3 women for every two men, that is, 3 women for every 5 players, that is, 60 women for every 100 players. Therefore, the proportion of women is 60% and the proportion of men is 40%.

Alternatively, from the equality$n_D = 1{,}5n_H$, we have:

$$p_D = 1{,}5p_H$$

And how $p_D + p_H = 1$, we have:

$$2{,}5p_H = 1 \quad \Rightarrow \quad p_H = \frac{1}{2{,}5} = 0{,}4 = 40\% \quad \Rightarrow \quad p_D = 1 - p_H = 1 - 0{,}4 = 0{,}6 = 60\%$$