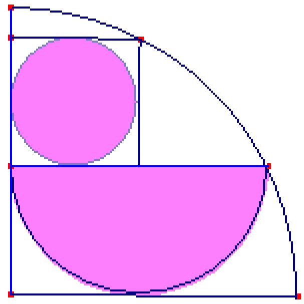
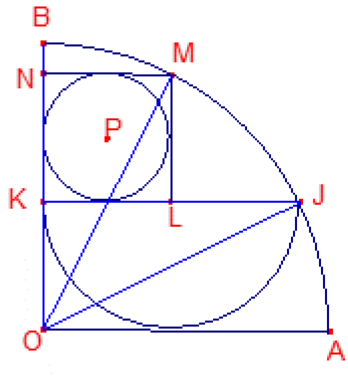
SOLUTIONS JUNE 2022

PROBLEMS USING GEOMETRIC PROGRAMS. AUTHOR: RICARD PEIRÓ i ESTRUCH. IES "Abastos". València

June 1: Calculate the ratio between the shaded area and the area of the quadrant.



Solution: Let be the quadrant $\widehat{AB}$ of centre O and radius $\overline{OA} = R$. Let be the semicircle with centre L and diameter $\overline{KJ} = 2r$. Let be the circle inscribed in the square KLMN of centre P. Let be $\overline{KN} = 2s$, s the radius of the circumference. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OKJ}$



$$5r^2 = R^2, \quad \overline{ON} = r + 2s, \overline{MN} = 2s, \overline{OM} = R$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ONM}$

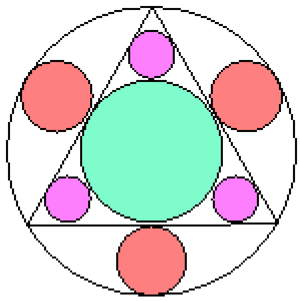
$$R^2 = 4r^2 + (r + 2s)^2; \; 10s^2 + \sqrt{5}Rs - R^2 = 0; \;\; s = \frac{\sqrt{5}}{10}R$$

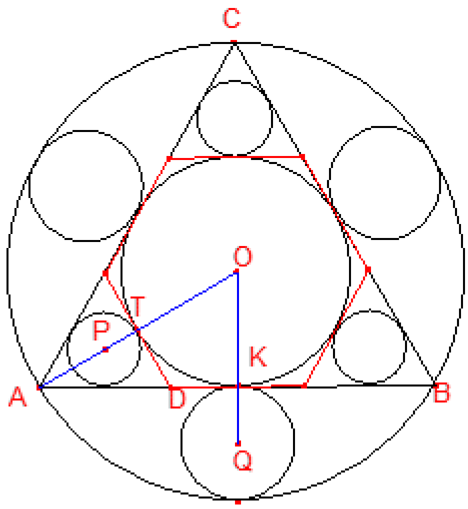
Let us note that r = 2s. The ratio of the areas is:

$$\frac{S_{shaded}}{S_{quadrant}} = \frac{\frac{1}{2}\pi r^2 + \pi s^2}{\frac{1}{4}R^2} = \frac{\frac{1}{2}\frac{1}{5} + \frac{1}{20}}{\frac{1}{4}} = \frac{3}{5}$$

June 2-3: An equilateral triangle is inscribed in a circle of radius R. 7 circles have been drawn. Calculate the radius of the circles.

Sangaku. Chiba Headquarters





Solution: Be the equilateral triangle $\overset{\Delta}{ABC}$. Let be the circle with centre O and radius $\overline{OA} = R$. Let be the inscribed circle in the equilateral triangle of radius $\overline{OT} = \overline{OK} = r$

$$\frac{r}{R} = \frac{1}{2}$$

Let T be the point of tangency. Let be the circle with centre P and radius $\overline{PT} = s$. Through point T we draw a parallel to the side $\overline{BC}$ that cuts the side $\overline{AB}$ in the point D.

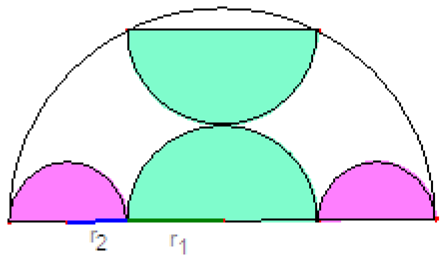
$$\overline{AD} = \frac{1}{3}\overline{AB}; \quad s = \frac{1}{3}r = \frac{1}{6}R$$

Let be the circle with centre Q and radius $\overline{QK} = t$

$$2R - 2t = \overline{CK} = 3r; \;\; 2t = 2R - 3r = 2R - \frac{3}{2}R = \frac{1}{2}R; \;\; t = \frac{1}{4}R$$

June 4: In the following figure, calculate:

$$\frac{r_1}{r_2}$$



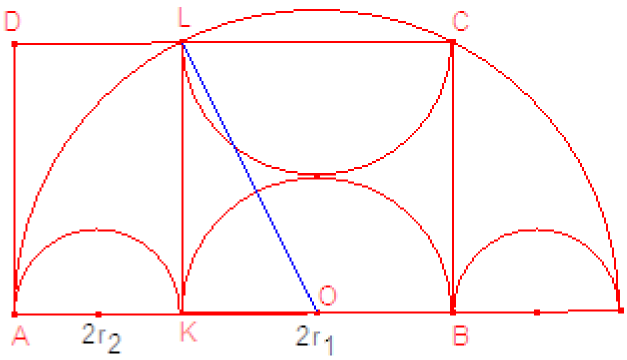
Solution 1: Let be the semicircle with centre O.

Diameters $\overline{BK}, \overline{CL}$ form a square.

$$\overline{OL} = \frac{\sqrt{5}}{2} r_1$$

Rectangle ABCD is golden.

$$\frac{\overline{AB}}{\overline{AD}} = \Phi = \frac{1+\sqrt{5}}{2}; \; \frac{r_1}{r_2} = \frac{\overline{KB}}{\overline{AK}} = \frac{\overline{AB}}{\overline{AD}} = \Phi$$
$$= \frac{1+\sqrt{5}}{2}$$



Solution 2: Let be the semicircle with centre O and radius $\overline{OL} = r_1 + 2 \cdot r_2$

$$\overline{OK} = r_1, \overline{KL} = 2 \cdot r_1$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OKL}$:

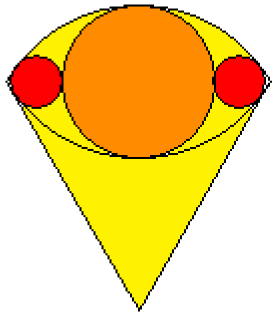
$$(r_1 + 2 \cdot r_2)^2 = {r_1}^2 + (2 \cdot r_1)^2$$

Simplifying:

$${r_1}^2 - r_2 \cdot r_1 - 1 = 0$$

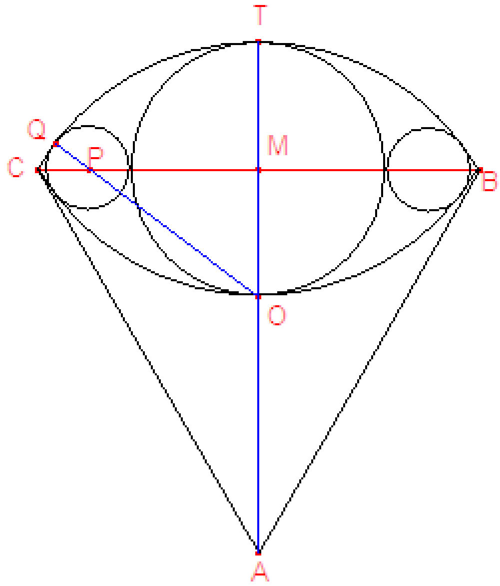
Solving the equation:

$$\frac{r_1}{r_2} = \frac{1+\sqrt{5}}{2}$$



June 6-7: In the following figure, determine the ratio between the radii of the two types of circles.

Sangaku. Tochigi Headquarters



Solution: Let be the triangle $\overset{\Delta}{ABC}$ and O the centre of the upper arch. Let M be the midpoint of the side $\overline{BC}$ centre of the large circle. Let be $\overline{OM} = R$ the radius of the large circle. Let P be the centre of the small circle on the left, of radius $s = \overline{PQ}$

$$\overline{OQ} = \overline{OA} = 2 \cdot \overline{MO} = 2R$$

$$\overline{PM} = R + s, \overline{OM} = R, \overline{OP} = \overline{OQ} - \overline{QP} = \overline{OT} - S = 2R - s$$

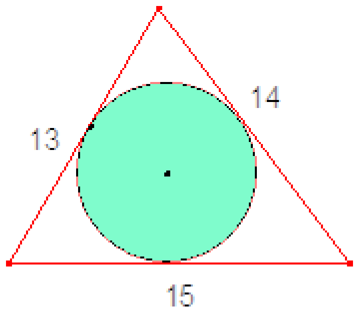
Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{OMP}$:

$$(2R - s)^2 = R^2 + (R + s)^2$$

Simplifying:

$$s = \frac{1}{3}R$$

June 8: The sides of a triangle measure 13, 14 and 15. Calculate the radius of the inscribed circle



Solution 1: Let r be the radius of the circle inscribed in the triangle. The area of the triangle is:

$$S = \frac{\sqrt{42 \cdot 16 \cdot 14 \cdot 12}}{4} = \frac{13 + 14 + 15}{2} r\,; \quad 84 = 21r$$

Solving the equation: r = 4. We notice that the triangle is heronian.

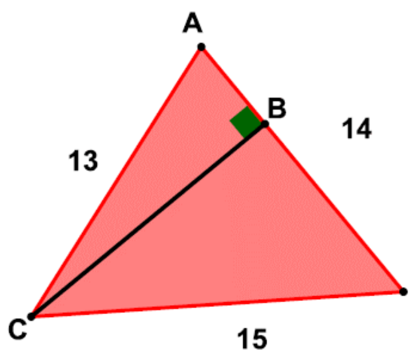
Solution 2: (Miguel Herrainz @M1GU3L_HH). We will have:

$$A = \frac{14 \cdot \overline{BC}}{2}$$

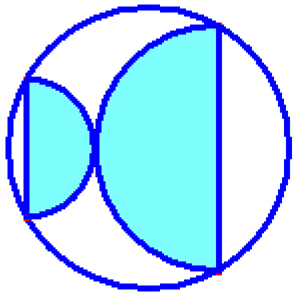
Remembering the Pythagorean triple 5-12-13, we will have:

$$\overline{CB} = 12; \ \ \overline{AB} = 5$$

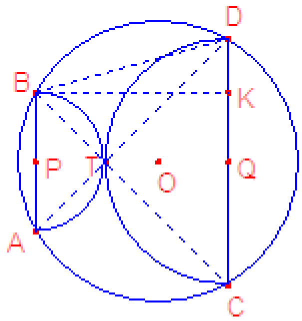
So:



$$A = \frac{14 \cdot 12}{2} = 84 = \frac{R(13+14+15)}{2} \Rightarrow R = 4$$



June 9: The diameters of the semicircles are parallel. Calculate the ratio between the shaded area and the area of the circle.



Solution: Let be the circle with centre O and radius R. Let be the semicircle with diameter $\overline{AB} = 2r$ and centre P Let be the semicircle of diameter $\overline{CD} = 2s$ and centre Q. Let T be the point of tangency of the two circumferences: The lines PT and AB are perpendicular. Lines PT and DC are perpendicular.

$$\angle ATB = \angle CTD = 90°\,; \quad \angle BAT = \angle CDT = 45°$$

So points A, T and D are aligned..

Let K be the projection of B on the line CD.

$$\overline{BK} = \overline{PQ} = r + s, \overline{DK} = |r - s|$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{BKD}$:

$$\overline{BD}^2 = (r+s)^2 + (r-s)^2 = 2r^2 + 2s^2; \ \overline{BD} = \sqrt{2(r^2+s^2)}; \ \ \overline{AT} = r\sqrt{2}, \overline{DT} = s\sqrt{2}, \overline{AD} = (r+s)\sqrt{2}$$

Applying the sinus theorem to the triangle $\overset{\Delta}{ABD}$:

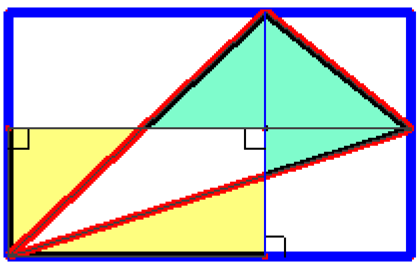
$$\frac{\sqrt{2(r^2+s^2)}}{\sin 45°} = 2R, \qquad \frac{\sqrt{2(r^2+s^2)}}{\frac{\sqrt{2}}{2}} = 2R$$

Simplifying:

$$R^2 = r^2 + s^2$$

The ratio of the areas is:

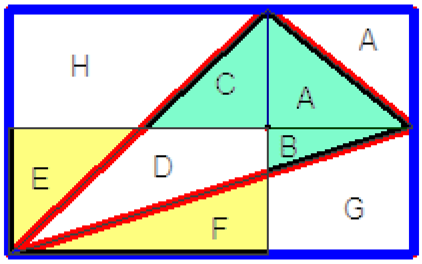
$$\frac{S_{shaded}}{S_{Total}} = \frac{\frac{1}{2}\pi r^2 + \frac{1}{2}\pi s^2}{\pi R^2} = \frac{1}{2}$$



June 10-11: The area of the red triangle is one third of the green outer rectangle. Calculate the ratio between the area painted green and the area painted yellow.

Solution: The red triangle and the blue rectangle and the two perpendicular segments divide the outer rectangle into 9 parts of areas:

$$A, A, B, C, D, E, F, G, H$$



The diagonal of a rectangle divides the rectangle into two equal parts.

$$H + E = C + D + F; \quad B + D + E = F + G$$

The area of the red triangle is the third part of the blue rectangle, so:

$$A + H + E + F + G = 2(A + B + C + D)$$

$$E + F + H + G = A + 2B + 2C + 2D$$

$$E + F + (C + D + F - E) + G = A + 2B + 2C + 2D$$

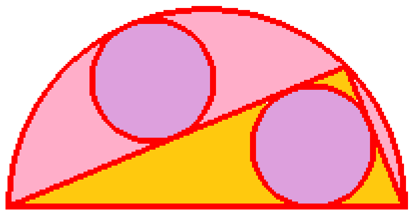
$$E + F + F - E + G = A + B + C + B + D$$

$$E + F + F - E + G = A + B + C + F + G - E$$

Then:

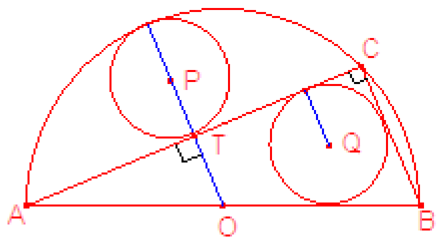
$$E + F = A + B + C$$

June 13: The two circles have radius 4. Calculate the radius of the semicircle.



Solution: Let be the semicircle with centre O and diameter $\overline{AB} = 2R$. Let be the right triangle $\overset{\Delta}{ABC}$, with $\angle C = 90°$. Let be the circle with centre P and radius $\overline{PT} = 4$. T is the midpoint of the side $\overline{AC}$, $\angle ATO = 90°$

$$\overline{OT} = R - 8$$



Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{AOT}$:

$$\overline{AT} = \sqrt{R^2 - (R - 8)^2} = \sqrt{16R - 64}$$

The right triangles $\overset{\Delta}{AOT}, \overset{\Delta}{ABC}$ are similar and of ratio $1:2$

$$\overline{BC} = 2R - 16, \overline{AC} = 2\sqrt{16R - 64}$$

The radius of the inscribed circle of the right triangle $\overset{\Delta}{ABC}$ is:

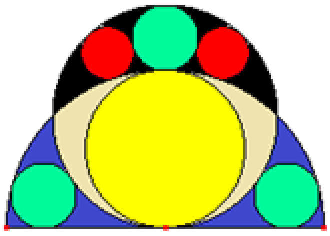
$$4 = r = \frac{\overline{AC} + \overline{BC} - \overline{AB}}{2} \quad ; \quad 4 = \frac{2\sqrt{16R - 64} + 2R - 16 - 2R}{2}$$

Simplifying:

$$12 = \sqrt{16R - 64}$$

Solving the equation:

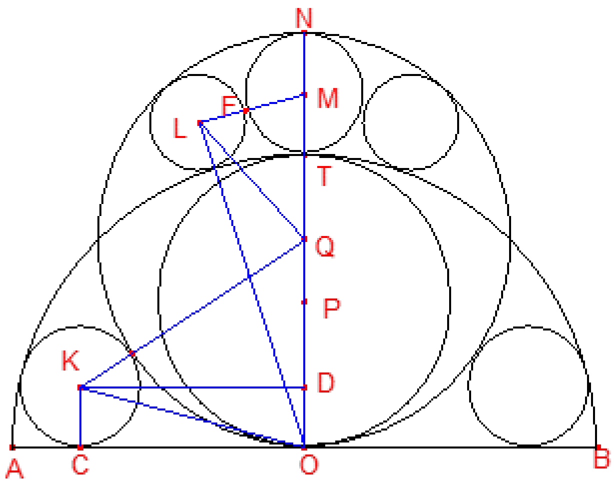
$$R = 13$$



June 14-15: In the figure, the radius of the semicircle is R = 1. Calculate the radius of the four types of circle.

Fukushima headquarters.

Solution: Let be the semicircle of diameter $\overline{AB} = 2$ and centre O. Let be the circle with centre P and diameter $\overline{OT} = 1$. With radius $\overline{PO} = \frac{1}{2}$. Let be the circle with centre K and radius $\overline{KC} = r$. Let be the circle with centre Q of diameter $\overline{ON} = 1 + 2r$



Let D be the projection of K onto $\overline{ON}$. Let be $\overline{KD} = a$.

$$\overline{OK} = 1 - r, \overline{QK} = \frac{1}{2} + 2r, \overline{QD} = \frac{1}{2}$$

Applying the Pythagorean theorem to right triangles $\overset{\Delta}{OCK}, \overset{\Delta}{KDQ}$:

$$a^2 = (1-r)^2 - r^2\ ;\ \ a^2 = \left(\tfrac{1}{2} + 2r\right)^2 - \left(\tfrac{1}{2}\right)^2$$

Matching the expressions:

$$(1-r)^2 - r^2 = \left(\frac{1}{2} + 2r\right)^2 - \left(\frac{1}{2}\right)^2$$

Simplifying and solving the equation:

$$4r^2 + 4r - 1 = 0, \qquad r = \frac{-1 + \sqrt{2}}{2}$$

The radius of the circumference of diameter $\overline{ON}$ is: $\overline{QO} = \frac{\sqrt{2}}{2}$. Let be the circle with centre L and radius $\overline{LF} = s$. Let be the circle with centre M and radius $\overline{MF} = \overline{MT} = r$.

$$\overline{OL} = 1 + s, \overline{LM} = r + s, \overline{QL} = \frac{\sqrt{2}}{2} - s, \overline{OM} = 1 + r, \overline{QM} = \frac{1}{2}$$

Let be $\angle LON = \alpha$. Applying the law of cosines to the triangle $\overset{\Delta}{LON}$:

$$(r+s)^2 = (1+s)^2 + (1+r)^2 - 2(1+s)(1+r)\cos\alpha\ ; \qquad \cos\alpha = \frac{1 + r + s - rs}{(1+s)(1+r)}$$

Applying the law of cosines to the triangle $\overset{\Delta}{LOQ}$:

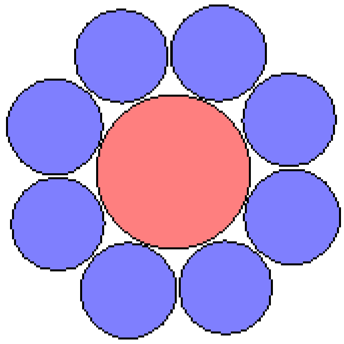
$$\left(\frac{\sqrt{2}}{2} - s\right)^2 = (1+s)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 - 2(1+s)\frac{\sqrt{2}}{2}\cos\alpha$$

$$\left(\frac{\sqrt{2}}{2} - s\right)^2 = (1+s)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 - 2(1+s)\frac{\sqrt{2}}{2}\frac{1 + r + s - rs}{(1+s)(1+r)}$$

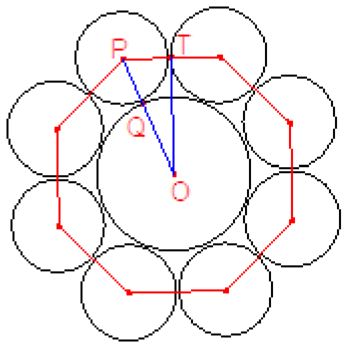
$$\sqrt{2}s = 1 + 2s - \frac{\frac{1+\sqrt{2}}{2} + s - \frac{-1+\sqrt{2}}{2}s}{\frac{1+\sqrt{2}}{2}}\sqrt{2}$$

Solving the equation:

$$s = \frac{1}{6}$$



June 16-17: Eight circles are exterior tangents two to two and all are exterior tangents to one another. You calculate the proportion between the two types of circumferences. Calculate the ratio between the areas of the sum of the eight blue and the red.



Solution: Let O be the centre of the red circle of radius $\overline{OQ} = r$. Let P be the centre of a blue circle of radius $\overline{PQ} = s$. Let T be the point of tangency of two blue circles. The centres of the eight blue circles are the vertices of a regular octagon.

$$\angle POT = \frac{1}{2}45°$$

Applying trigonometric ratios to the right triangle $\overset{\Delta}{OTP}$

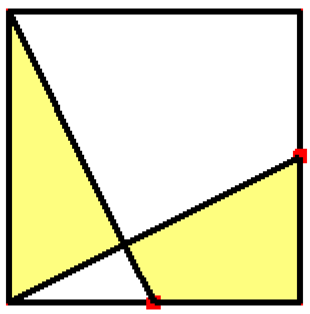
$$\frac{s}{r+s} = \sin\frac{45°}{2} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

Solving the equation, the ratio between the radii:

$$\frac{s}{r} = \frac{\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}} \approx 0.6199$$

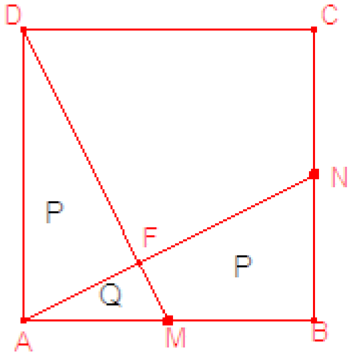
The ratio between the areas of the sum of the eight blue and the red is:

$$\frac{S_{blue}}{S_{red}} = 8 \cdot \left(\frac{s}{r}\right)^2 = 8\frac{2-\sqrt{2}}{\left(2-\sqrt{2-\sqrt{2}}\right)^2} = 3.0744$$



June 18: The indicated points are midpoints of the sides of the square. Calculate the ratio of the areas of the shaded region and the square.

Solution: Let be the square ABCD. Let M and N be the midpoints of the sides $\overline{AB}, \overline{BC}$, respectively. Let F be the intersection of $\overline{DM}$ and $\overline{AN}$. Let S be the area of square ABCD. Let P be the area of the right triangle. $\overset{\Delta}{AFD}$. Let Q be the area of the right triangle. $\overset{\Delta}{AFM}$.



$$P + Q = \frac{1}{4}S$$

The right triangles $\overset{\Delta}{AFD}, \overset{\Delta}{MFA}$ are similar and of ratio 2:1.

Their areas are proportional to the square of the ratio of the sides.:

$$P = 4 \cdot Q$$

Then:

$$Q = \frac{1}{20}S, P = \frac{1}{5}S$$

With the shaded region we have:

$$S_{shaded} = 2P = 2 \cdot \frac{1}{5}S = \frac{2}{5}S$$

The ratio between the areas of the shaded area and the square is:

$$\frac{S_{shaded}}{S} = \frac{2}{5}$$

June 20-21: Nine circles tangent two to two are inside another circle. Calculate the proportion between the areas of the sum of the nine circumferences and the outer circumference.

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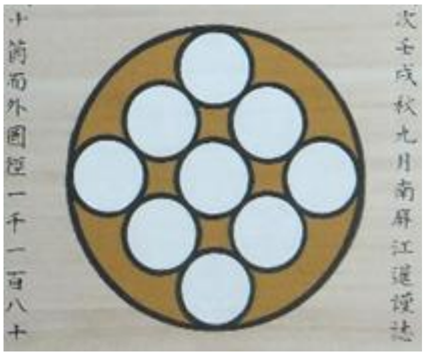
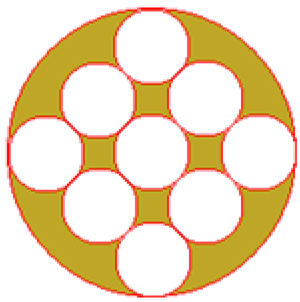
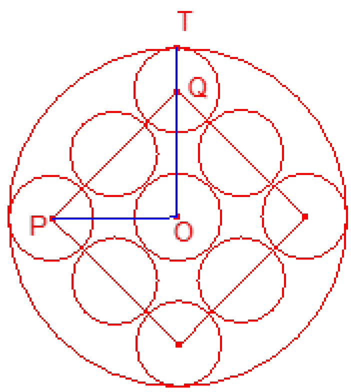




Solution: Let be the outer circle with centre O and radius $\overline{OT} = R$. Let be the circle with centre Q and radius $\overline{QT} = r$. Let P be the centre of the circle and radius r.



$$\overline{PQ} = 4r$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{POQ}$:

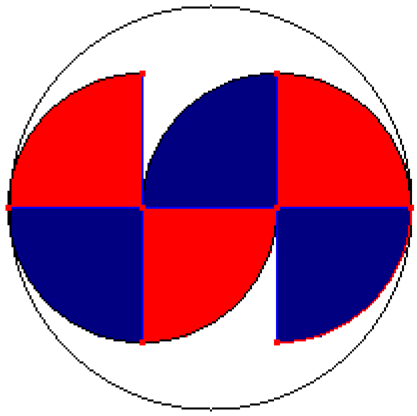
$$\overline{OQ} = 2r\sqrt{2}$$

So:

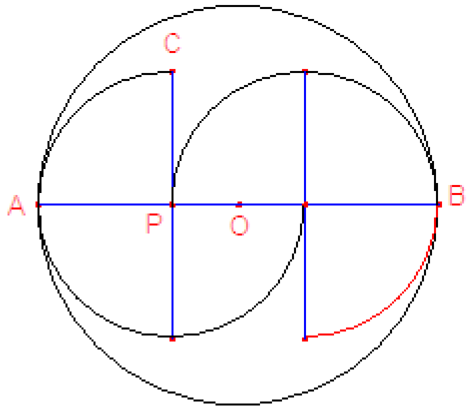
$$R = (1 + 2\sqrt{2})r$$

The ratio of the areas of the nine circumferences to the outer circumference is.

$$\frac{9 \cdot S_Q}{S_O} = 9\left(\frac{r}{R}\right)^2 = 9\frac{1}{(1 + 2\sqrt{2})^2} = \frac{9(9 - 4\sqrt{2})}{49} \approx 0.6140$$



<u>June 22:</u> Calculate the ratio between the area of the shaded area and the area of the outer circle.



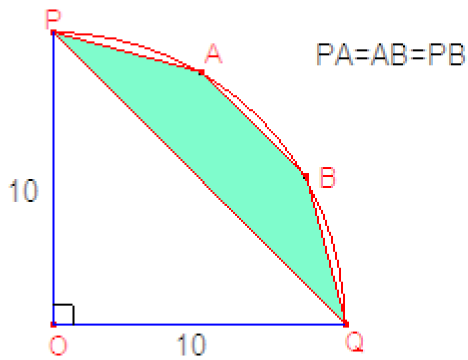
<u>Solution:</u> Let P be the centre of the square of circumference. Let $\overline{PA} = r$ be the radius of the quadrant of the circumference. Let $\overline{AB} = 3r$ be the diameter of the outer circle with centre O. The radius of the outer circle is:

$$\overline{OA} = \frac{3}{2}r$$

The ratio of areas is:

$$\frac{S_{shaded}}{S_{total}} = \frac{\frac{3}{2} \cdot r^2}{R^2} = \frac{\frac{3}{2} \cdot r^2}{\left(\frac{3}{2}r\right)^2} = \frac{2}{3}$$

<u>June 23:</u> Let be the quadrant with centre O and radius $\overline{OP} = \overline{OQ} = 10$. Let A and B be the points of the arc such that $\overline{PA} = \overline{AB} = \overline{PB}$. Calculate the area of quadrilateral PABQ.



<u>Solution:</u>

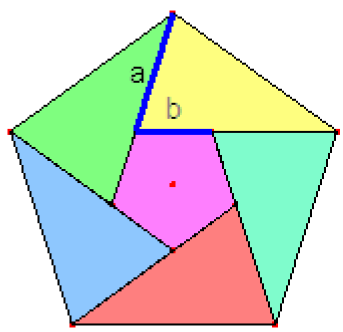
$$\angle POA = \angle AOB = \angle BOQ = 30°$$

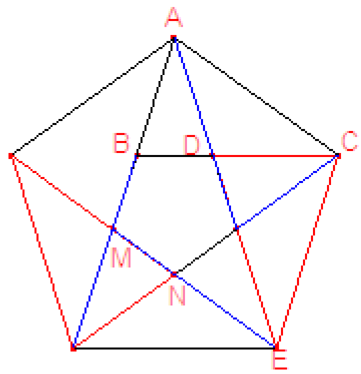
The area of quadrilateral PABQ is:

$$\begin{aligned} S_{PABQ} &= 3 \cdot S_{POA} - S_{OPQ} \\ &= 3 \cdot \frac{1}{2} \cdot 10 \cdot 10 \cdot \sin 30° - \frac{1}{2} \cdot 10 \cdot 10 \\ &= 75 - 50 = 25 \end{aligned}$$

<u>June 24-25:</u> The regular pentagon in the figure has been divided into five triangles and one regular pentagon. ~~All six regions have the same area~~. Calculate.

$$\frac{a}{b}$$





Note 1: The phrase in red and crossed out must be eliminated from the statement, as it is false and does not apply to the demonstration of the problem. see later notes.

Solution: Let be the triangle $\overset{\Delta}{ABC}$, with $\overline{AB} = a$. Let D be on side $\overline{BC}$ such that $\overline{BD} = b, \overline{BC} = a + b$. A, B and M are aligned. So, M, N and E are aligned, $\overline{EM} = \overline{BC} = a + b$, $\angle BMN = 108°, \angle ABC = 72°$, $\angle MAE = 36°$

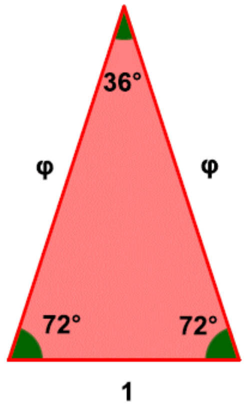
Then the triangle $\overset{\Delta}{ABC}$ is isosceles and golden.

So:

$$\frac{a}{b} = \Phi = \frac{1 + \sqrt{5}}{2}$$

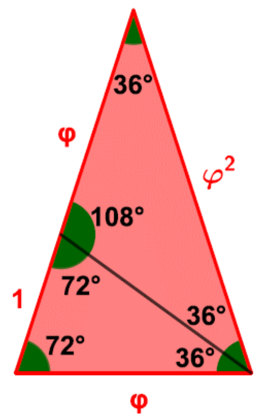
Note that the areas of the two regular pentagons are in proportion $1:5$

Note 2: A 72°-36°-72° triangle is called a golden triangle. If in it, the unequal side measures 1 we have that the equal sides measure:



$$\varphi = \frac{1 + \sqrt{5}}{2} = \Phi$$

Every golden triangle breaks down into a golden triangle and a golden gnomon (triangle 36°-108°-36°). The attached figure shows the decomposition of the golden triangle with sides $\varphi^2 - \varphi - \varphi^2$ because in it a very important equality is demonstrated, namely:



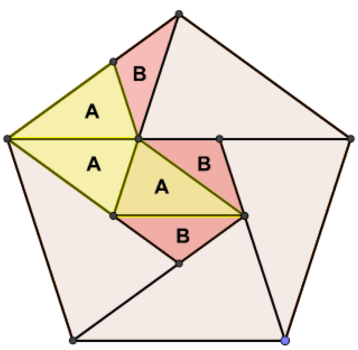
$$1 + \varphi = \varphi^2$$

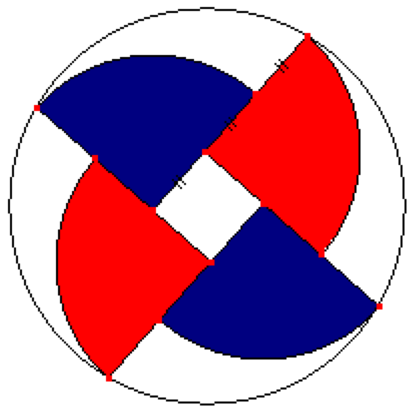
in the golden triangle φ-1-φ, It is true that:

$$\text{Area} = \frac{(1 + \varphi)\sqrt{3 - \varphi}}{4} \text{ ; Perimeter} = 1 + 2\varphi = \varphi(1 + \varphi)$$

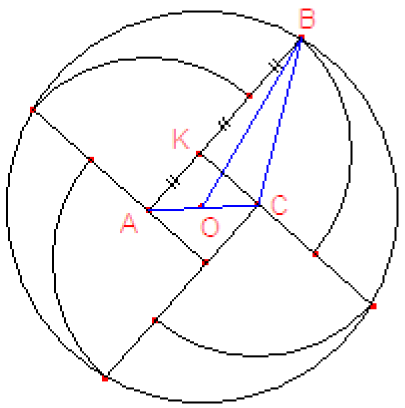
$$\cos 36° = \frac{\varphi}{2}$$

Note 3: The sentence removed from the statement is never fulfilled, since the area of the inner pentagon is never equal to the area of any of the five associated triangles next to the initial pentagon: Each of these triangles breaks down into two golden triangles of area A and a gnomon golden, with area B, while the inner pentagon breaks down into a golden triangle and two golden gnomons. If the two polygons had the same area, it should be true that 2.A + B = A + 2B, which would lead to A = B, which is obviously not true.





June 27: Calculate the ratio between the area of the shaded area and the area of the outer circle.



Solution: Let be the quadrant with centre K and radius $\overline{KB} = 2a$. Let A and C be centres of two quadrants. Let $\overline{AK} = a$. The centre O of the outer circle is the midpoint of the segment $\overline{AC}$.Let $\overline{OB} = R$ be the radius of the outer circumference. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{AKC}$

$$\overline{AC} = a\sqrt{2}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{BKC}$

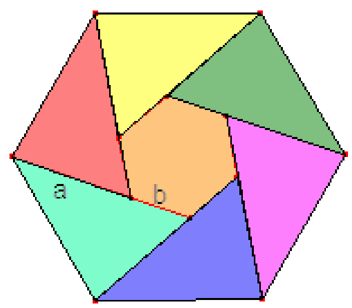
$$\overline{BC} = a\sqrt{5}$$

$\overline{OB}$ is the median of the triangle $\overset{\Delta}{ABC}$, then:

$$R^2 = \frac{2 \cdot \left(a\sqrt{5}\right)^2 - 2 \cdot (3a)^2 - \left(a\sqrt{5}\right)^2}{4}, \qquad R^2 = \frac{13}{2}a^2$$

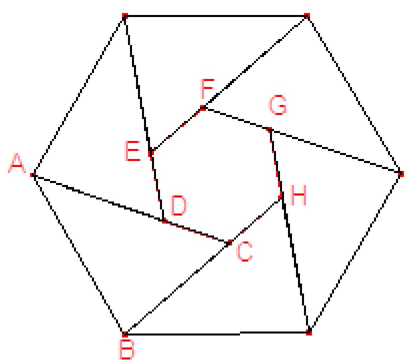
The ratio of areas is:

$$\frac{S_{saded}}{S_{total}} = \frac{(2a)^2}{R^2} = \frac{8}{13}$$



June 28-29: All seven regions in the figure have the same area. Calculate

$$\frac{a}{b}$$



Solution: Let $\overline{AD} = \overline{BC} = a, \overline{CD} = b$; $\angle BCA = 60°$. The area of the triangle $\overset{\Delta}{ABC}$ and the area of the regular hexagon CDEFGH are equal.

$$\frac{1}{2}(a+b)a\frac{\sqrt{3}}{2} = 6\frac{\sqrt{3}}{4}b^2$$

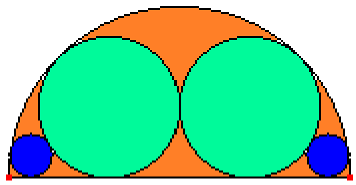
Simplifying:

$$a^2 + ab - 6b^2 = 0$$

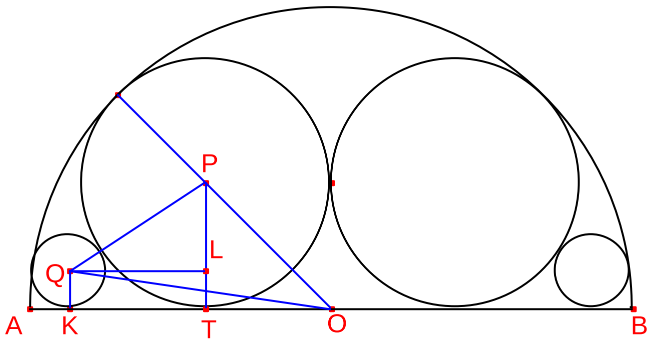
Solving the equation:

$$\frac{a}{b} = 2$$

<u>June 30:</u> Given the semicircle of radius R, calculate the radii of the other circles.



<u>Solution:</u>



Let be the semicircle with centre O and diameter $\overline{AB} = 2R$. Let be the large circle with centre P and radius $\overline{OT} = \overline{OT} = r$.

$$\overline{OP} = R - r = r\sqrt{2}$$

Then:

$$r = (\sqrt{2} - 1)R$$

Let be the small circle with centre Q and radius $\overline{QK} = s$. Let L be the projection of Q onto $\overline{PT}$

$$\overline{PQ} = r + s, \overline{PL} = r - s$$

Let $\overline{QL} = a$. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{QLP}$

$$a^2 = 4rs$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{QKO}$

$$(R - s)^2 = (a + r)^2 + s^2$$

Simplifying:

$$(2\sqrt{2} - 1)s + (1 - \sqrt{2})R + 2(\sqrt{2} - 1)\sqrt{(\sqrt{2} - 1)Rs}$$

Solving the equation:

$$s = \frac{5\sqrt{2} - 1}{49}R$$